

Trabajo Fin de Máster
Máster Universitario en Ingeniería de Caminos,
Canales y Puertos

A 2.5D spectral approach based on the BEM
and the FEM to study wave propagation in
fluid acoustics and elastodynamics

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Dep. Mecánica de Medios Continuos y Teoría de
Estructuras
Escuela Técnica Superior de Ingeniería
Universidad de Sevilla

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El tribunal nombrado para juzgar el trabajo arriba indicado, compuesto por los siguientes profesores:

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Resumen

En este trabajo se presenta una formulación espectral en dos dimensiones y media (2.5D) basada en el método de los elementos finitos (MEF) y el método de los elementos de contorno (MEC) para estudiar la propagación de ondas en medios acústicos y elásticos en problemas tridimensionales (3D). El análisis se lleva a cabo mediante la superposición de problemas bidimensionales (2D) para diferentes números de onda a lo largo de la coordenada longitudinal. El método numérico está basado en la descomposición del dominio para representar el sistema fluido-estructura. Se presenta en este trabajo un elemento finito espectral para representar ondas guiadas en sólidos con sección transversal arbitraria. Además, el MEC se extiende a la formulación espectral para estudiar medios fluidos infinitos. Ambas aproximaciones utilizan polinomios de interpolación de Lagrange como funciones de forma en los puntos de Legendre-Gauss-Lobatto (LGL). Las técnicas propuestas se han verificado mediante dos problemas de referencia: la propagación de ondas en un medio fluido infinito en presencia de una cavidad rígida fija y la propagación de ondas en un carril libre, y los resultados obtenidos presentan un buen grado de acuerdo con las soluciones de referencia. Se realiza un breve análisis $h - p$ para evaluar la precisión de los métodos.

Abstract

This work presents a two-and-a-half dimensional (2.5D) spectral formulation based on the finite element method (FEM) and the boundary element method (BEM) to study three dimensional (3D) wave propagation in fluid acoustics and elastodynamics. The analysis is carried out by superposing two dimensional (2D) problems for different longitudinal wavenumbers. The numerical method is based on the domain decomposition to study structure-fluid system. A spectral finite element to represent waveguides in solids with arbitrary cross-section is proposed. Moreover, the BEM is extended to its spectral formulation to study unbounded fluid-acoustics media. Both approaches use Lagrange interpolant polynomials as element shape functions at the Legendre-Gauss-Lobatto (LGL) points. The proposed techniques are verified from two benchmark problems: the wave propagation in an unbounded fluid medium in presence of a fixed rigid cavity and the wave propagation in a free rail. The computed results are in good agreement with the reference solution. A brief $h-p$ analysis is done to assess the accuracy of the methods.

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1 Introduction

1.1 Problem outline and motivation

Time-harmonic wave propagation, such as fluid acoustics and solid scattering, is a common phenomenon that appears in many engineering fields. Hybrid methods based on the Boundary Element Method (BEM) and the Finite Element Method (FEM) are suitable to study solid and fluid interaction in unbounded media. The propagation of acoustic waves triggered by static and moving pressure sources, the vibration assessment and the acoustic insulation involve fluid and solid interaction and must be considered rigorously. The finite element method (FEM) have been used in several works to predict the response in fluid-structure interaction problems. For the low frequency range, the conventional finite elements with linear shape represent accurately the fluid and solid scattering waves. However, at high frequencies, these shape functions do not provide reliable results due to so-called pollution effects [1, 14]: the accuracy of the numerical solution deteriorates with increasing non-dimensional wave number and it is not sufficient the commonly employed rules of n elements per wavelength [11]. High element resolutions are required in order to obtain results with reasonable accuracy.

The pollution effect (numerical dispersion) cannot be avoided in two and three dimensional problems [17], but it can be reduced by using higher order shape functions. Ihlenburg and Babuška [12, 13] demonstrates that the numerical dispersion of FEM for Helmholtz equation is related to the stability of the Helmholtz differential operator at high wavenumber. Moreover, high-order numerical formulation improves the accuracy of the numerical method in order to reduce the dimensions of the discretisation. Deraemaeker et al. [8] studied several finite element methodologies for the Helmholtz equation. They concluded that the p -FEM drastically reduces the pollution error. Mehdizadeh and Paraschivoiu [16] compared a Spectral Element Method (SEM) and a second-order FEM. The authors concluded that the two-dimensional SEM leads to fewer number of elements per wavelength and less computational cost, in terms of both memory and computational time, for the same accuracy. Vos et al. [23] presented an efficient implementation of a spectral/ $h - p$ element method and they analysed which specific combination of mesh size h and polynomial order p minimises the computational cost to solve a problem up to a predefined level of accuracy for smooth and non-smooth solutions. In the $h - p$ method the integration is usually done by numerical integration which can induce numerical error [2].

The SEM is close to the p -FEM [18], and handles some of the advantages regarding the representation of non-homogeneity properties. The studied domain is discretized into elements and the field variable is usually represented with high-order interpolation shape function through the integration points. The p -FEM and spectral methodologies have been developed using several shape function families: Lagrange, Legendre and Bernstein polynomials. Biermann et al. [5] and Petersen et al. [19] studied several shape functions families for acoustic simulations. They concluded regarding the simulation of acoustic phenomena, governed by the Helmholtz equation, that the high-order finite elements are an efficient method even in the higher frequency range, and the elements based on Bernstein polynomials provided the most efficient and stable solution procedure for the analysed problems. Bériot et al. [4] compared several polynomial bases commonly used in FEM and SEM for controlling the pollution and to lead to more efficient technique for convected wave propagation problems. The authors concluded that the properties of p -FEM that make its strength for standard acoustics remain for flow acoustics as well. At high wave numbers, where the dispersion error was

likely to dominate the total numerical error, p -FEM elements proved to efficiently mitigate the pollution effect also for convected applications.

Many authors have successfully applied the spectral element method in different fields. Degrande and De Roeck [7] developed a high-order method for studying the dynamic response of a porous saturated medium using spectral elements. They use as shape functions, in a displacement formulation, the analytical solution for Biot equations restricted to two-dimensional (2D) plane strain wave propagation. Bar-Yoseph et al. [3] presented a space-time spectral element method to solve a simply supported modified Euler-Bernoulli beam undergoing external forced lateral vibrations. They examined two Galerkin formulations using Hermitian and Lagrangian polynomials as interpolants for spatial and time discretisations. The space-time SEM allows the use of larger time steps while maintaining the accuracy of the solution. Kudela et al. [15] applied spectral finite elements for studying the wave propagation in an isotropic rod and in a Timoshenko beam. The authors proposed this methodology to detect small damage in structures. The computed results were compared to those obtained from the classical FEM and from measurements. This comparison highlighted the efficiency of the spectral elements and it confirms that the methodology reflects this phenomenon quite well. Zhu et al. [24] coupled Chebyshev spectral elements with the implicit Newmark time integral method to simulate acoustic field. The presented results indicated that the accuracy increased as the time step decreased. The result of this study also concluded that uniform distribution of node is better for the improvement of numerical accuracy.

1.2 Objectives

The method proposed in this work regards with a two-and-a-half dimensional (2.5D) approach to represent fluid-solid interaction (FSI) problems. The study of three-dimensional (3D) scattered waves in fluid and solid media requires the use of highly computational demanding models. The solution becomes much simpler if the medium is homogeneous in one direction, even if the dynamic source remains 3D. Such situation is referred to as a two-and-a-half dimensional problems [6]. This work proposes a 2.5D spectral formulation based on the FEM and the BEM to study wave propagation in fluid-structure interaction problems. The proposed approach is useful for problems where the material and geometric properties are uniform along one direction, and the source exhibits 3D behaviour.

1.3 Original contributions

The following points give an outline of what are perceived to be the significant contributions from the present work.

- The boundary element method in 2.5D in fluid acoustics has been extended to its spectral formulation to allow for the computation of wave propagation problems in fluid media. This introduces a new accurate method, with lower computational efforts, to solve problems at a higher frequency range.
- Moreover, a new 2.5D spectral finite element is presented to study solid waveguides.

1.4 Organization of the text

The outline of this work is as follows. First, the numerical model is presented. The spectral finite elements formulation in elastodynamics is described. Then, the spectral boundary element method in fluid acoustics is discussed. Finally, the proposed methods are then verified with two benchmark problems [20, 10].

2 Numerical model

The spectral formulation is based on a 2.5D coupled BEM-FEM formulation, where the solid (Ω_s) is represented by the FEM, while the limiting interface (Γ_f) between the solid and the fluid ($\Omega_{f\infty}$) is modelled with the BEM (Figure 2.1). In this work both formulations have been developed separately and the coupling procedure is a line of a future research.

The 2.5D formulation computes the problem solution as the superposition of two-dimensional (2D) problems with a different longitudinal wavenumber, k_z , in the z direction. An inverse Fourier transform is used to compute the 3D solution:

$$a(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} \hat{a}(\hat{\mathbf{x}}, k_z, \omega) e^{-ik_z z} dk_z \quad (2.1)$$

where $a(\mathbf{x}, \omega)$ is the unknown magnitude (e.g., displacement or pressure), $\hat{a}(\hat{\mathbf{x}}, k_z, \omega)$ is its representation in the frequency-wavenumber domain, $\hat{\mathbf{x}} = \mathbf{x}(x, y, 0)$, ω is the angular frequency, and $i = \sqrt{-1}$. The integral defined by Equation (2.1) is transformed into a summation as follows:

$$a(\mathbf{x}, \omega) = \sum_{-\infty}^{+\infty} \hat{a}(\hat{\mathbf{x}}, k_z, \omega) e^{-ik_z z} \Delta k_z \quad (2.2)$$

with $k_z = m\Delta k_z$. This equation converges and can be approximated by a finite sum of terms.

2.1 The 2.5D spectral boundary element formulation

The boundary element formulation presented in this work considers an arbitrary boundary submerged in an unbounded fluid medium. The integral representation of the pressure p^i for a point i located at the fluid subdomain $\Omega_{f\infty}$, with zero body forces and zero initial conditions may be written as [9]:

$$c^i p^i(\mathbf{x}^i, \omega) = \int_{\Gamma_f} p^{i*}(\mathbf{x}, \omega; \mathbf{x}^i) u^i(\mathbf{x}, \omega) d\Gamma - \int_{\Gamma_f} u^{i*}(\mathbf{x}, \omega; \mathbf{x}^i) p^i(\mathbf{x}, \omega) d\Gamma \quad (2.3)$$

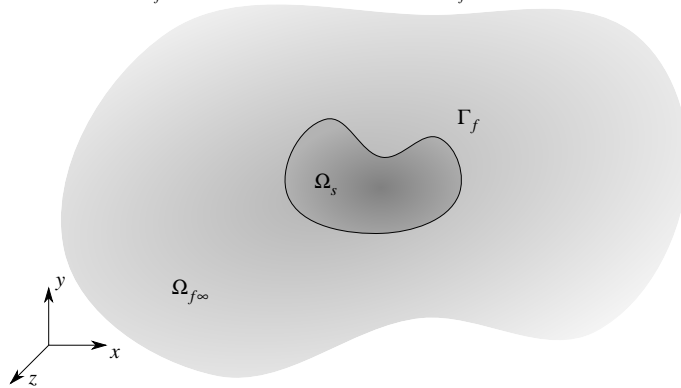


Figure 2.1 Domain decomposition.

where $u^i(\mathbf{x}, \omega)$ and $p^i(\mathbf{x}, \omega)$ are respectively the normal displacement to boundary Γ_f and the nodal pressure. $u^{i*}(\mathbf{x}, \omega; \mathbf{x}^i)$ and $p^{i*}(\mathbf{x}, \omega; \mathbf{x}^i)$ are respectively the fluid full-space fundamental solution for normal displacement and pressure at point \mathbf{x} due to a point load at \mathbf{x}^i . The integral-free term c^i depends only on the boundary geometry at point i . The integration boundary Γ_f represents the boundary between the unbounded fluid medium ($\Omega_{f\infty}$) and the solid subdomain (Ω_s).

Assuming that the unbounded medium is constant in the longitudinal direction z , Equation (2.3) is expressed in terms of integrals in this direction and over the cross-section boundary, Σ_f :

$$c^i p^i(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} \int_{\Sigma_f} p^{i*}(\mathbf{x}, \omega; \mathbf{x}^i) u^i(\mathbf{x}, \omega) dS dz - \int_{-\infty}^{+\infty} \int_{\Sigma_f} u^{i*}(\mathbf{x}, \omega; \mathbf{x}^i) p^i(\mathbf{x}, \omega) dS dz \quad (2.4)$$

Equation (2.4) is then transformed to the wavenumber domain as:

$$c^i \hat{p}^i(\hat{\mathbf{x}}, \omega, k_z) = \int_{\Sigma_f} \hat{p}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i) \hat{u}^i(\hat{\mathbf{x}}, \omega, k_z) dS - \int_{\Sigma_f} \hat{u}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i) \hat{p}^i(\hat{\mathbf{x}}, \omega, k_z) dS \quad (2.5)$$

where a hat above a variable denotes its representation in the frequency-wavenumber domain.

The problem is discretised into elements, leading to a boundary approximation of the normal displacement and pressure using the interpolation shape functions ϕ^j . Then, Equation (2.5) is written as:

$$c^i \hat{p}^i = \sum_{j=1}^Q \left[\left\{ \int_{\Sigma_f^j} \hat{p}^{i*} \phi^j d\Sigma \right\} \hat{u}^i - \left\{ \int_{\Sigma_f^j} \hat{u}^{i*} \phi^j d\Sigma \right\} \hat{p}^i \right] = \sum_{j=1}^Q [\hat{\mathbf{H}}^{ij} \hat{u}^i - \hat{\mathbf{G}}^{ij} \hat{p}^i] \quad (2.6)$$

where Q is the number of boundary nodes at the boundary Σ_f and Σ_f^j are the elements which contains the node j . After interpolating the boundary variables, the integral representation defined by Equation (2.6) yields a system of equations that is solved for each frequency.

The spatial integrations in Equation (2.6) are numerically evaluated by a Lobatto-Gauss-Legendre (LGL) quadrature when the collocation point does not belong to the integration element. In a first approach, a LGL quadrature is also used to evaluate the singular integrals when the collocation point belongs to the integration element using an appropriate number of integration points. References [21, 22] state that the use of a sufficient number of quadrature points leads to a good approximation of the singular integrals.

The system of equations for all the boundary elements becomes:

$$\hat{\mathbf{H}}(\omega, k_z) \hat{\mathbf{u}}(\hat{\mathbf{x}}, \omega, k_z) = \hat{\mathbf{G}}(\omega, k_z) \hat{\mathbf{p}}(\hat{\mathbf{x}}, \omega, k_z) \quad (2.7)$$

The proposed spectral boundary element method for the 2.5D fluid element uses Legendre polynomials of order p as interpolation shape functions. The shape interpolation functions ϕ are given by:

$$\phi_i = \prod_{j \neq i} \frac{\xi - \xi_j}{\xi_j - \xi_i} \quad (2.8)$$

where the local nodal coordinates ξ are found at the LGL integration points:

$$(1 - \xi^2) \frac{\partial \phi(\xi)}{\partial \xi} = 0 \quad (2.9)$$

The element integration is carried out using the LGL points. Then, the computation of matrices $\hat{\mathbf{H}}$ y $\hat{\mathbf{G}}$ is simplified as follows:

$$\begin{aligned} \hat{h}_{ij} &= \hat{p}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i) w_j |\mathbf{J}(\xi_j)| \\ \hat{g}_{ij} &= \hat{u}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i) w_j |\mathbf{J}(\xi_j)| \end{aligned} \quad (2.10)$$

where w_j is the LGL integration weight at the natural nodal coordinate ξ_j , and $|\mathbf{J}(\xi_j)|$ is the Jacobian evaluated at ξ_j , $\hat{u}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i)$ and $\hat{p}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i)$ are respectively normal displacement to boundary Γ_f and nodal pressure, and both are derived from a potential:

$$\begin{aligned} \hat{u}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i) &= -\frac{i}{4} H_1^{(2)}(k_\alpha r) \frac{\partial r}{\partial \mathbf{n}} \\ \hat{p}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i) &= \frac{i}{4} H_0^{(2)}(k_\alpha r) \end{aligned} \quad (2.11)$$

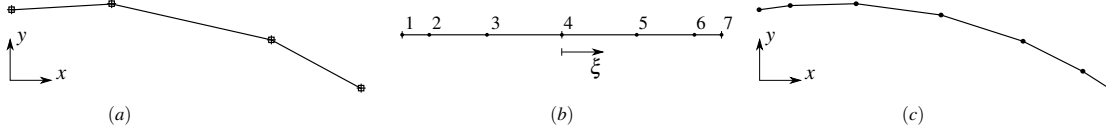


Figure 2.2 (a) Element mesh seed representation, (b) element definition and (c) physical representation of spectral fluid boundary element of order $p = 6$.

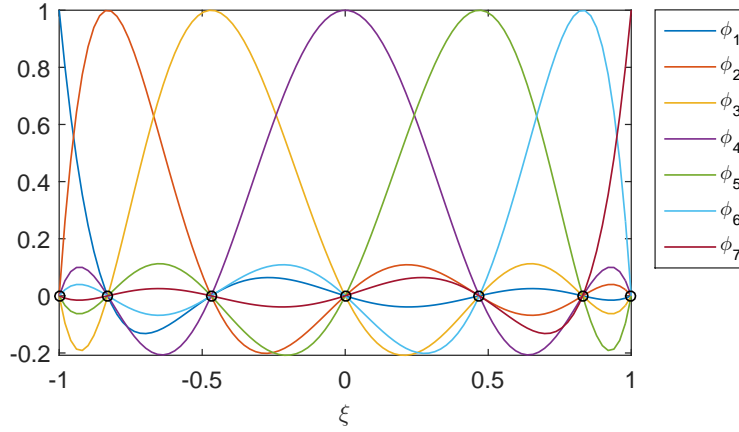


Figure 2.3 Element shape functions for an element with order $p = 6$.

where $H_n^{(2)}$ are second Hankel functions of order n , $k_\alpha = \sqrt{\frac{\omega^2}{\alpha^2} - k_z^2}$, with $\text{Im}(k_\alpha) < 0$, α is the sound propagation velocity and $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

The boundary Γ_f is discretized into elements that are generated from a base mesh. The element definition is done by using a polynomial interpolation from the element base mesh and its representation in the natural coordinates system. As an example, Figure 2.2.(a) shows an element base mesh defined by four nodes used to generate an element with order $p = 6$ (Figure 2.2.(c)). The spectral element nodes are defined in natural coordinates in ascending order (Figure 2.2.(b)) with $\xi \in [-1, 1]$.

The shape function ϕ and its derivatives are symbolically computed in straightforward procedure for an arbitrary element order. Figure 2.3 shows the interpolation shape functions for the $p = 6$ spectral element presented in Figure 2.2.

2.2 The 2.5D spectral finite element formulation

The spectral formulation for solid subdomains is based on the virtual work principle. For any virtual displacement field $\delta \mathbf{u}$ imposed on the solid subdomain Ω_s , the virtual work due to the internal and the inertial forces must be equal to the virtual work produced by the external loads [25]:

$$\begin{aligned} & -\omega^2 \int_{\Omega_s} \delta \mathbf{u}(\mathbf{x}, \omega) \rho_s \mathbf{u}(\mathbf{x}, \omega) d\Omega + \int_{\Omega_s} \delta \boldsymbol{\varepsilon}(\mathbf{x}, \omega) \boldsymbol{\sigma}(\mathbf{x}, \omega) d\Omega \\ & = \int_{\Omega_s} \delta \mathbf{u}(\mathbf{x}, \omega) \rho_s \mathbf{b}(\mathbf{x}, \omega) d\Omega + \int_{\Gamma_s} \delta \mathbf{u}(\mathbf{x}, \omega) \mathbf{f}(\mathbf{x}, \omega) d\Gamma \end{aligned} \quad (2.12)$$

where $\mathbf{u}(\mathbf{x}, \omega)$ is the displacement vector, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are respectively the strain and stress tensors, $\rho_s \mathbf{b}(\mathbf{x}, \omega)$ is the body force in the domain Ω_s , ρ_s is the solid density, and $\mathbf{f}(\mathbf{x}, \omega)$ is the nodal force vector. A variable preceded by δ denotes a virtual change of its magnitude.

The Voigt notation is used to write both the symmetrical stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}\}^T$ and $\boldsymbol{\varepsilon} = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\}^T$, respectively. The use of engineering shear strains allows to write the internal work as an inner product $\boldsymbol{\varepsilon}_b^T \boldsymbol{\sigma}_b$.

The displacement vector $\mathbf{u}(\mathbf{x}, \omega)$ is discretized as:

$$\mathbf{u}(\mathbf{x}, \omega) = \mathbf{N}\mathbf{\underline{u}}(\mathbf{x}, \omega) \quad (2.13)$$

where \mathbf{N} are the globally defined shape functions and $\mathbf{\underline{u}}(\mathbf{x}, \omega)$ is the nodal displacement vector.

The strain tensor $\boldsymbol{\varepsilon}$ is derived from the nodal displacement vector $\mathbf{\underline{u}}(\mathbf{x}, \omega)$ as:

$$\boldsymbol{\varepsilon} = \mathbf{L}_1 \mathbf{N} \mathbf{\underline{u}} + \mathbf{L}_2 \mathbf{N} \frac{\partial \mathbf{\underline{u}}}{\partial z} \quad (2.14)$$

where

$$\mathbf{L}_1 = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial x} \end{bmatrix} \quad (2.15)$$

and

$$\mathbf{L}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (2.16)$$

The strain vector $\boldsymbol{\varepsilon}$ in Equation (2.14) is related to both the displacement vector $\mathbf{\underline{u}}$ and its derivative $\partial \mathbf{\underline{u}} / \partial z$ with respect to the longitudinal coordinate z . This differs from the relationship $\boldsymbol{\varepsilon} = \mathbf{L} \mathbf{N} \mathbf{\underline{u}}$ for the 3D formulation, where the strain vector is written as a linear combination of the elements of the nodal displacement vector $\mathbf{\underline{u}}$. Equation (2.14) is alternatively written as:

$$\boldsymbol{\varepsilon} = \mathbf{B}_1 \mathbf{\underline{u}} + \mathbf{B}_2 \frac{\partial \mathbf{\underline{u}}}{\partial z} \quad (2.17)$$

where $\mathbf{B}_1 = \mathbf{L}_1 \mathbf{N}$ and $\mathbf{B}_2 = \mathbf{L}_2 \mathbf{N}$.

The stress tensor is related to the strain vector through the constitutive relation:

$$\boldsymbol{\sigma} = \mathbf{D} \boldsymbol{\varepsilon} \quad (2.18)$$

where, in the case of a linear isotropic material, the constitutive matrix \mathbf{D} depends on the Young's modulus E and the Poisson's ratio ν :

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad (2.19)$$

A standard Galerkin procedure is followed, using the same approximation for the virtual displacement vector. Substituting the strain-displacement relation defined by Equation (2.17), and the constitutive Equation (2.18) into the virtual work Equation (2.12) yields:

$$\begin{aligned} & -\omega^2 \int_{\Omega_s} \delta \mathbf{\underline{u}}^T \mathbf{N}^T \rho_s \mathbf{N} \mathbf{\underline{u}} d\Omega + \int_{\Omega_s} \left(\delta \mathbf{\underline{u}}^T \mathbf{B}_1^T + \delta \left(\frac{\partial \mathbf{\underline{u}}}{\partial z} \right)^T \mathbf{B}_2^T \right) \mathbf{D} \left(\mathbf{B}_1 \mathbf{\underline{u}} + \mathbf{B}_2 \frac{\partial \mathbf{\underline{u}}}{\partial z} \right) d\Omega \\ & = \int_{\Omega_b} \delta \mathbf{\underline{u}}^T \mathbf{N}^T \rho_s \mathbf{b} d\Omega + \int_{\Gamma_f} \delta \mathbf{\underline{u}}^T \mathbf{N}^T \mathbf{f} d\Gamma \end{aligned} \quad (2.20)$$

Equation (2.20) is further elaborated, rewriting the volume integrals over the longitudinal coordinate z and the cross-section A_s :

$$\begin{aligned}
& -\omega^2 \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{N}^T \rho_s \mathbf{N} dA \right) \underline{\mathbf{u}} dz + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 dA \right) \underline{\mathbf{u}} dz \\
& + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 dA \right) \frac{\partial \underline{\mathbf{u}}}{\partial z} dz + \int_{-\infty}^{\infty} \delta \left(\frac{\partial \underline{\mathbf{u}}}{\partial z} \right)^T \left(\int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 dA \right) \underline{\mathbf{u}} dz \\
& + \int_{-\infty}^{\infty} \delta \left(\frac{\partial \underline{\mathbf{u}}}{\partial z} \right)^T \left(\int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 dA \right) \frac{\partial \underline{\mathbf{u}}}{\partial z} dz \\
& = \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{N}^T \rho_s \mathbf{b} dA \right) dz + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{\Sigma_f} \mathbf{N}^T \mathbf{f} d\Sigma \right) dz
\end{aligned} \tag{2.21}$$

where Σ_f is the intersection of the surface Γ_f with the plane $z = 0$ (Figure 2.1). The discretized equation is obtained through elimination of the virtual displacement vector $\delta \underline{\mathbf{u}}$. This requires integration by parts on the terms containing derivatives $\delta (\partial \underline{\mathbf{u}} / \partial z)$ in Equation (2.21):

$$\begin{aligned}
& -\omega^2 \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left[\int_{A_s} \mathbf{N}^T \rho_s \mathbf{N} dA \right] \underline{\mathbf{u}} dz + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left[\int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 dA \right] \underline{\mathbf{u}} dz \\
& + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left[\int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 dA - \int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 dA \right] \frac{\partial \underline{\mathbf{u}}}{\partial z} dz \\
& - \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 dA \right) \frac{\partial^2 \underline{\mathbf{u}}}{\partial z^2} dz \\
& = \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{N}^T \rho_s \mathbf{b} dA \right) dz + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{\Sigma_f} \mathbf{N}^T \mathbf{f} d\Sigma \right) dz
\end{aligned} \tag{2.22}$$

Since Equation (2.22) holds for any virtual displacement $\delta \underline{\mathbf{u}}$, all integrals over the longitudinal coordinate z vanish and Equation (2.22) is written as follows:

$$-\omega^2 \mathbf{M} \underline{\mathbf{u}} + \mathbf{K}^0 \underline{\mathbf{u}} + \mathbf{K}^1 \frac{\partial \underline{\mathbf{u}}}{\partial z} + \mathbf{K}^2 \frac{\partial^2 \underline{\mathbf{u}}}{\partial z^2} = \underline{\mathbf{f}} \tag{2.23}$$

where the mass matrix \mathbf{M} is defined as:

$$\mathbf{M} = \int_{A_s} \mathbf{N}^T \rho_s \mathbf{N} dA \tag{2.24}$$

and stiffness matrices \mathbf{K}^0 , \mathbf{K}^1 and \mathbf{K}^2 are defined as:

$$\begin{aligned}
\mathbf{K}^0 &= \int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 dA \\
\mathbf{K}^1 &= \int_{A_s} (\mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 - \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1) dA \\
\mathbf{K}^2 &= \int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 dA
\end{aligned} \tag{2.25}$$

The external load vector:

$$\underline{\mathbf{f}} = \int_{A_s} \mathbf{N}^T \rho_s \mathbf{b}(x, y, z, \omega) dA + \int_{\Sigma_f} \mathbf{N}^T \mathbf{f}(x, y, z, \omega) d\Sigma \tag{2.26}$$

contains contributions for both body forces and surface loads and it is evaluated for every point z along the longitudinal axis.

The differential Equation (2.23) is solved by a Fourier transform of the longitudinal coordinate z to the horizontal wavenumber k_z , where the Fourier transform is defined as $\mathcal{F}[f(z), k_z] = \int_{-\infty}^{\infty} \exp(+ik_z z) f(z) dz$. Equation (2.23) is transformed to the wavenumber domain as:

$$[-\omega^2 \mathbf{M} + \mathbf{K}^0 - ik_z \mathbf{K}^1 - k_z^2 \mathbf{K}^2] \hat{\underline{\mathbf{u}}}(k_z, \omega) = \hat{\underline{\mathbf{f}}}(k_z, \omega) \tag{2.27}$$

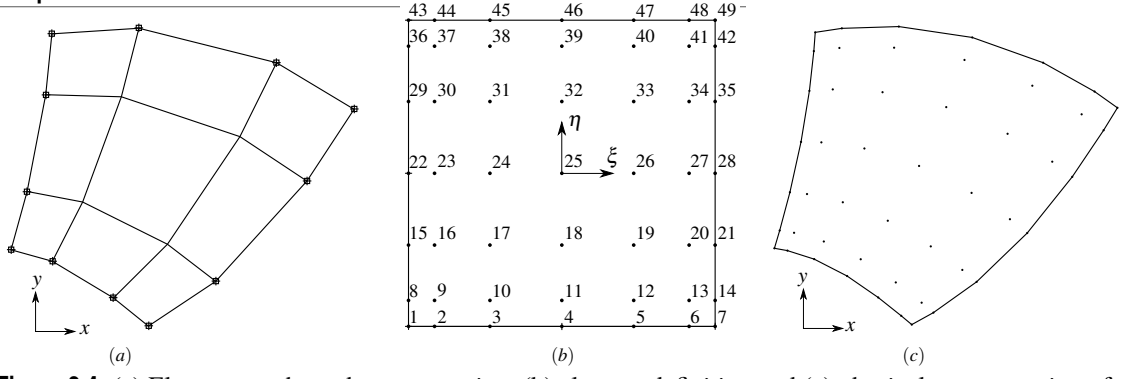


Figure 2.4 (a) Element mesh seed representation, (b) element definition and (c) physical representation of a solid spectral element of order $p = 6$.

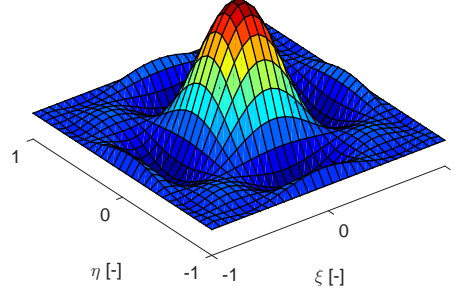


Figure 2.5 Element shape function of $p = 6$ N_{25} for solid finite element.

where a hat above a variable denotes its representation in the frequency-wavenumber domain.

Equation (2.27) is rewritten if an equivalent dynamic stiffness matrix $\hat{\mathbf{K}}$ is considered:

$$\hat{\mathbf{K}}(k_z, \omega) \hat{\mathbf{u}}(\hat{\mathbf{x}}, k_z, \omega) = \hat{\mathbf{f}}(\hat{\mathbf{x}}, k_z, \omega) \quad (2.28)$$

where the dynamic stiffness matrix $\hat{\mathbf{K}}$ is easily computed for each wavenumber and frequency step.

The proposed spectral finite element method uses the 2D Legendre polynomials of order p as specific shape functions \mathbf{N} , and the local nodal element coordinates $\boldsymbol{\xi}(\xi, \eta) \in [-1, 1] \times [-1, 1]$ are defined as the Lobatto-Gauss-Legendre (LGL) points:

$$\begin{cases} (1 - \xi^2) \frac{\partial \mathbf{N}(\xi, \eta)}{\partial \xi} = 0 \\ (1 - \eta^2) \frac{\partial \mathbf{N}(\xi, \eta)}{\partial \eta} = 0 \end{cases} \quad (2.29)$$

Then, the local nodal element coordinate matches the LGL integration points, and defines an orthogonal element shape function basis. Therefore, the solid element matrices defined by Equations (2.24) and (2.25) are easily computed as follows:

$$\begin{aligned} m_{ij} &= \delta_{ij} \rho w_{ij} |\mathbf{J}(\xi_i, \eta_j)| \\ k_{ij}^2 &= \delta_{ij} \mathbf{L}_2^T \mathbf{D} \mathbf{L}_2 w_{ij} |\mathbf{J}(\xi_i, \eta_j)| \end{aligned} \quad (2.30)$$

where δ_{ij} is the Kronecker delta, w_{ij} is the LGL weight at the nodal coordinate system (ξ_i, η_j) , and $|\mathbf{J}(\xi_i, \eta_j)|$ is the Jacobian evaluated at (ξ_i, η_j) .

The solid subdomain Ω_s is discretized into elements that are generated from a base element mesh. The element definition is done by using a polynomial interpolation from the starting element geometry and its representation in the natural coordinates system. As an example, Figure 2.4. shows an element with order $p = 6$. The element nodes are defined in natural coordinates in ascending order (Figure 2.4.(b)) with $\boldsymbol{\xi}(\xi, \eta) \in [-1, 1] \times [-1, 1]$. The spectral solid element has three degrees of freedom at each node, $\hat{\mathbf{u}}(\hat{\mathbf{x}}, k_z, \omega) = \{\hat{u}_x(\hat{\mathbf{x}}, k_z, \omega), \hat{u}_y(\hat{\mathbf{x}}, k_z, \omega), \hat{u}_z(\hat{\mathbf{x}}, k_z, \omega)\}$.

The element shape functions are computed as $\mathbf{N}_k = \boldsymbol{\phi}_i(\xi) \boldsymbol{\phi}_j(\eta)$, where k is the element node in which the shape function takes unity value, $k = (i - 1)(p + 1) + j$, and $\boldsymbol{\phi}_i$ is the one-dimensional LGL shape function defined by Equation (2.8). The shape function \mathbf{N}_k and its derivatives are symbolically computed in straightforward procedure for an arbitrary element order. Figure 2.5 shows the interpolation shape function N_{25} for an spectral element with order $p = 6$ (Figure 2.4.(b)).

3 Numerical verification

The proposed spectral boundary and finite element methods were verified with two benchmark problems regarding to a fixed cylindrical inclusion in an unbounded fluid medium and the propagation of elastic waves in free rail.

3.1 Fixed cylindrical inclusion

The BEM model was verified with a benchmark problem. The model was implemented and validated by applying it to a fixed cylindrical circular cavity, submerged in a homogeneous unboundend fluid medium. The cavity is subjected to a harmonic point pressure load. The analytical solution to this problem can be found in Reference [20].

The cavity has a radius $r = 5$ m, and the unbounded fluid medium properties are pressure wave velocity $\alpha = 1500$ m/s and density $\rho = 1000$ kg/m³.

The problem was solved for a frequency range varying from 2.5 Hz to 320 Hz, with a frequency step $\Delta f = 2.5$ Hz. A constant value of $k_z = 10^{-5}$ rad/m was assumed. The problem solution was computed directly from Equation (2.7), and then the radiated wave field at receivers was assessed using the Somigliana identity [9].

The problem solution was computed for a dilatational point source placed at the fluid medium $\hat{\mathbf{x}}_0 = (x_0, y_0) = (0, 15)$ 15 m away from the cavity centre. This loads emits a harmonic incident field \hat{p}_{inc} at a point $\hat{\mathbf{x}}$ described by:

$$\hat{p}_{inc} = (\hat{\mathbf{x}}, \omega, k_z) = \frac{-iA}{2} H_0^{(2)}(k_\alpha \sqrt{(x - x_0)^2 + (y - y_0)^2}) \quad (3.1)$$

where A is the source amplitude, $H_n^{(2)}$ are the second Hankel functions of order n and $k_\alpha = \sqrt{\frac{\omega^2}{\alpha^2} - k_z^2}$ is the effective wavenumber ($\text{Im}\{k_p\} = 0$).

The element size and the approximation order was set to keep the nodal density per wavelength at least $d_\lambda = 12$ [4]. The characteristic problem wavelength was $\lambda_{min} = \alpha/f_{max} = 4.69$ m, so to accomplish the above condition the minimum number of nodes must be 80. Three different boundary discretization were investigated for characteristic element distribution $1/h = 2/\pi$ m⁻¹, $1/h = 3/\pi$ m⁻¹ and $1/h = 4/\pi$ m⁻¹. In this study, the element order is $p = 5$. As an example, Figure 3.1 shows a boundary discretization using $1/h = 2/\pi$ m⁻¹ and $p = 5$.

The scattered wave field was computed over a grid of receivers as shown in Figure 3.2.

Figure 3.3 shows the amplitude of the analytical solution of the scattered pressure field when acts a harmonic pressure load of 200 Hz and the total incident fields. The maximum amplitude was found near to the source point, and a shadowed region appeared after the cavity.

Figure 3.4 represents the difference between the analytical solution and computed results when the inclusion is modelled with a different number of boundary elements. Maximum errors occur in the shadowed zone and around the surface inclusion. The improvement of the computed solution as the distance to the surface of the cavity increases is noticeable. As expected, the boundary element method accuracy improves as element size decreases. The results showed that higher order elements or small element size must be used to compute the solution at points close to the cavity boundary, where the scattered field changes rapidly.

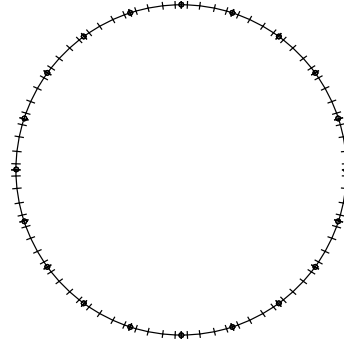


Figure 3.1 Cylindrical cavity embedded in an unbounded fluid medium: problem discretization with element distribution $1/h = 2/\pi$ and order $p = 6$.

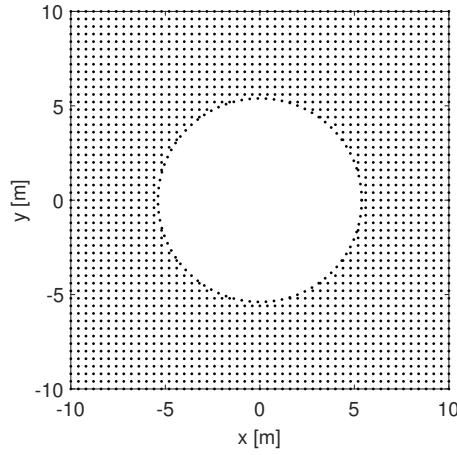


Figure 3.2 Receivers grid.

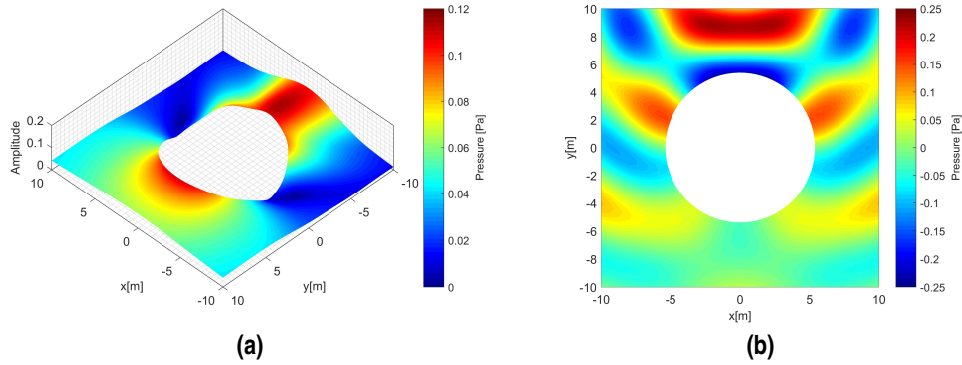


Figure 3.3 (a) Scattered pressure field and (b) total pressure field due to a harmonic pressure load acting at 200 Hz.

3.2 Propagative waves in a free rail

The solid spectral element presented in Section 2.1 was verified through computing of propagative waves in a free rail UIC861-3. This problem was previously studied by L. Gavric [10]. The dispersion curves and the cross-section modes were computed for the wavenumber range $0 - 35$ rad/m, with a wavenumber step $\Delta k_z = 0.05$ rad/m.

First, the problem was solved using linear finite elements with two different discretizations. The first mesh consists of 227 nodes and 176 finite elements, which results in a mesh of 681 degrees of freedom (Figure 3.5.(a)). The element size varies from $h = 5.7$ mm to $h = 8.6$ mm. This mesh is similar to the mesh used in

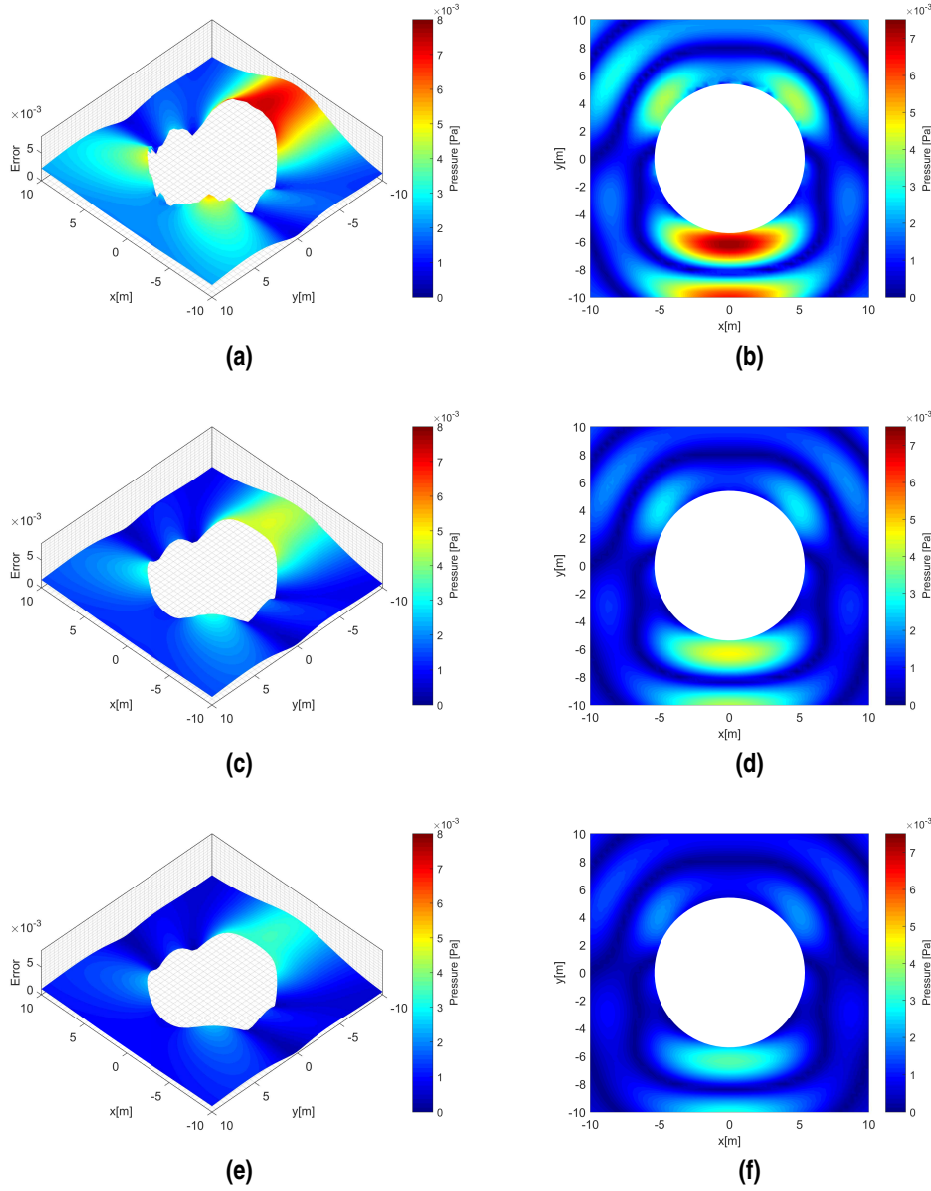


Figure 3.4 Absolute error obtained from discretizations (a-b) $1/h = 2/\pi \text{ m}^{-1}$, (c-d) $1/h = 3/\pi \text{ m}^{-1}$ and (f-g) $1/h = 4/\pi \text{ m}^{-1}$.

Reference [10]. The element size in the second case varies from $h = 2.9 \text{ mm}$ to $h = 4.3 \text{ mm}$, which means 805 nodes and 704 elements with 2415 degrees of freedom (Figure 3.5.(b)).

The computed results are compared with the reference solution [10]. Eight characteristic cross-sectional modes shapes were computed in the frequency range $0 - 6 \text{ kHz}$. Four modes had axial/vertical displacement componets (Figure 3.6), while the rest were characterized by dominant lateral in plane vibrations of the cross-section (Figure 3.7). For the low frequency range, the computed results show a good agreement with reference solution, regardless of elements size. Some differences appear between numerical results and reference results for higher frequencies.

Furthermore, an analysis using spectral finite elements has been made. The cross-section has been represented by seven elements, three elements to model the railhead, three elements to model the foot and one element to model the neck. This is an advantage of the SEM, which allows to discretize the domain with fewer elements than the FEM. The problem was solved using different element order to assess the accuracy. Figure 3.8 shows the mesh with element order $p = 4$ and $p = 8$.

Computed results show a good agreement for most of the previously described waves. The lateral waves

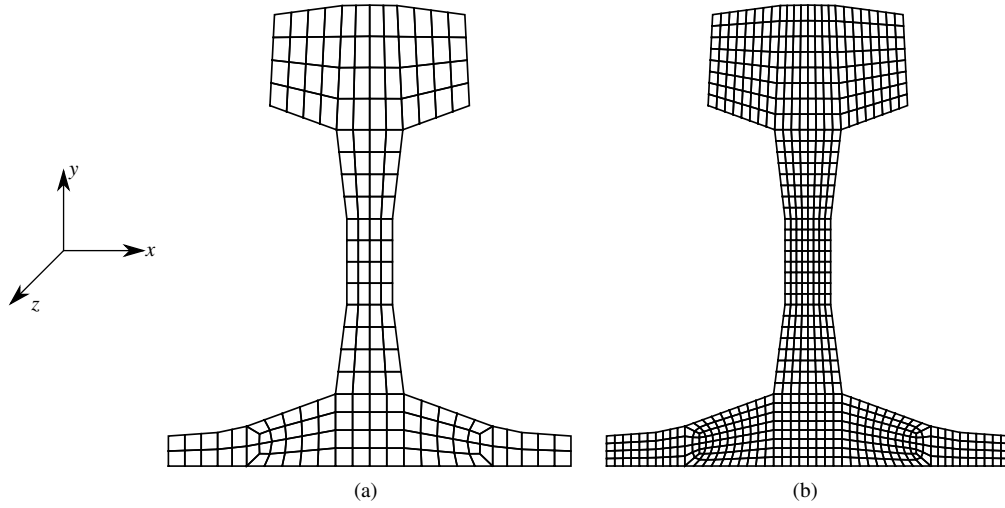


Figure 3.5 (a) Rail discretization for linear element size $h = 5.7 - 8.6$ mm elements and (b) $h = 2.9 - 4.3$ mm.

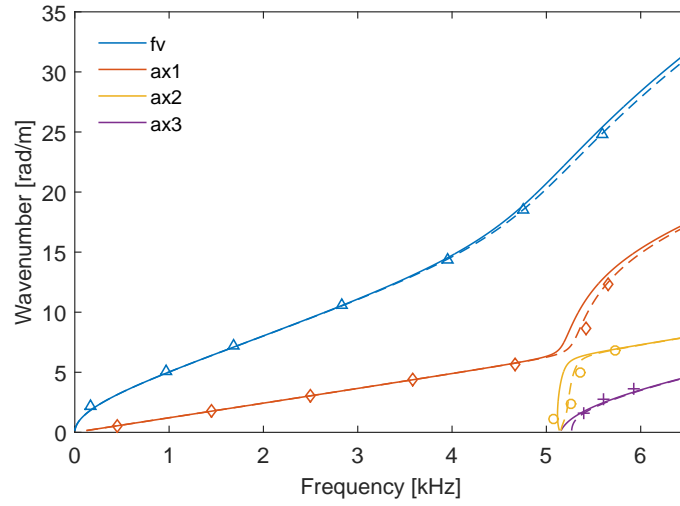


Figure 3.6 Dispersion curves of propagative vertical/longitudinal modes for the frequency range 0-6000 Hz. —, computed results for $h = 2.9 - 4.3$ mm; - -, computed results for $h = 5.7 - 8.6$ mm; \triangle , \diamond , \circ , +, reference results [10].

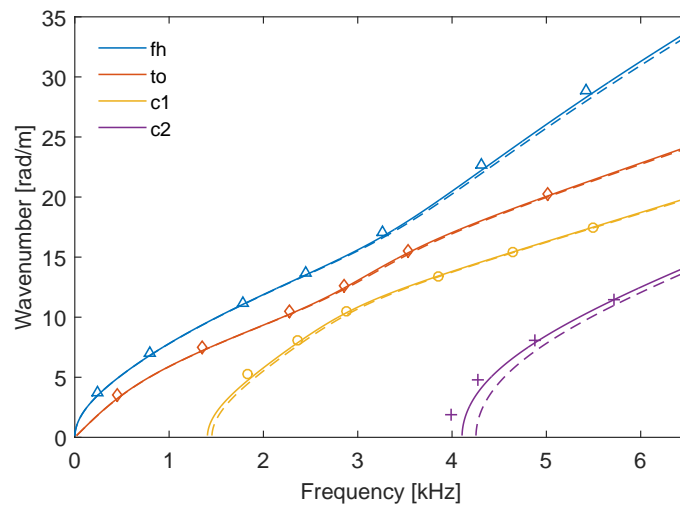


Figure 3.7 Dispersion curves of propagative lateral modes for the frequency range 0-6000 Hz. —, computed results for $h = 2.9 - 4.3$ mm; - -, computed results for $h = 5.7 - 8.6$ mm; \triangle , \diamond , \circ , +, reference results [10].

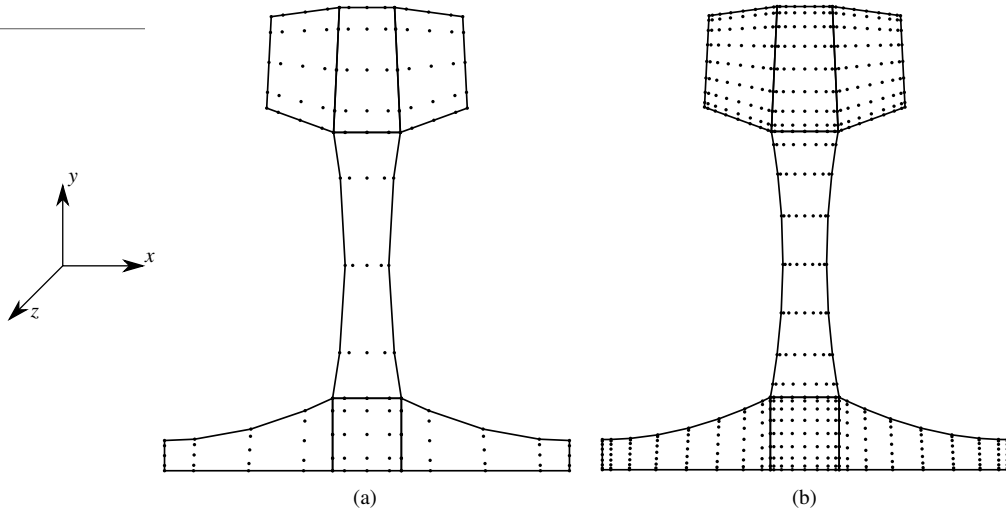


Figure 3.8 Rail discretization with spectral elements. Element order (a) $p = 4$ and (b) $p = 8$.

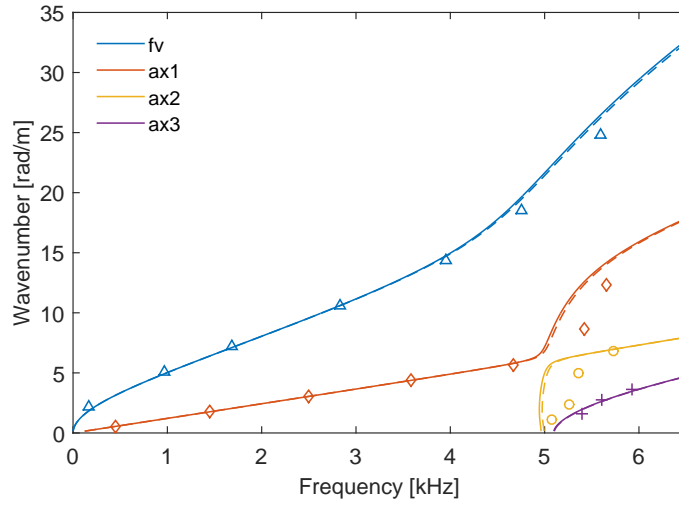


Figure 3.9 Dispersion curves of propagative vertical/longitudinal modes for the frequency range 0-6000 Hz. —, computed results for $p = 8$; - -, computed results for $p = 4$; \triangle , \diamond , \circ , $+$, reference results [10].

are quite similar to reference solution. Otherwise, axial curves denoted by $ax1$, $ax2$ and $ax3$ show appreciable differences with results presented in Reference [10] for the high frequencies. These differences could be explained by the coarse mesh used in Reference [10] for high frequencies. The proposed results in this work suggest that the solution for finer discretization converges to the SEM solution shown in Figures 3.9 and 3.10.

3.2.1 Axial waves in free rail

Figure 3.11 represents the displacement along the z axis due to the axial waves denoted by $ax1$, $ax2$ and $ax3$ for an excitation frequency of 5600 Hz. Despite these waves are axial in their basic nature, the dominant movement of the cross-section at high frequencies is characterized foot flapping. The third axial wave $ax3$ presents a characteristic S-shape deformation of the cross-section.

3.2.2 Vertical bending waves of the free rail

Figure 3.12 shows the displacement along the y axis due to the vertical flexural wave fv . At low frequencies, this type of wave corresponds to simple beam flexion characterized by vertical translation of the undeformed cross-section during the wave propagation. As the frequency increases, the foot flapping becomes predominant.

3.2.3 Horizontal flexion of the free rail

Figure 3.13 shows the displacement along the x axis due to the horizontal flexural wave fh . Although is referred as horizontal flexion, it involves a small rotation of the cross-section, even at a very low frequency. This is due to the non-symmetry on the x axis of the rail profile. At low frequencies, the cross-section mainly

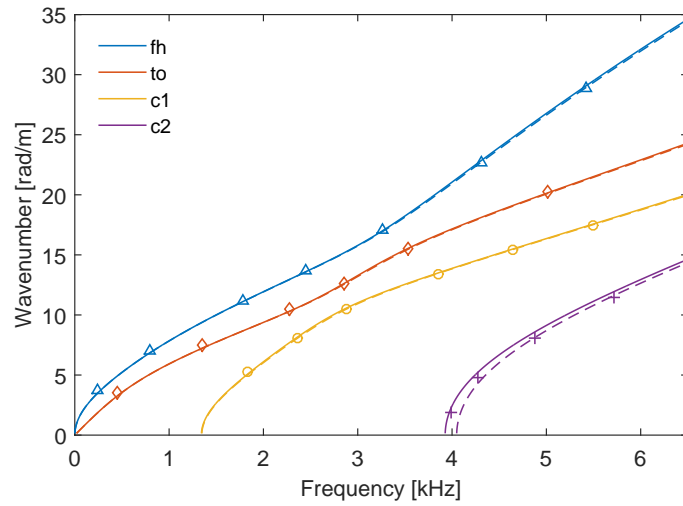


Figure 3.10 Dispersion curves of propagative lateral modes for the frequency range 0-6000 Hz. —, computed results for $p = 8$; - -, computed results for $p = 4$; \triangle , \diamond , \circ , $+$, reference results [10].

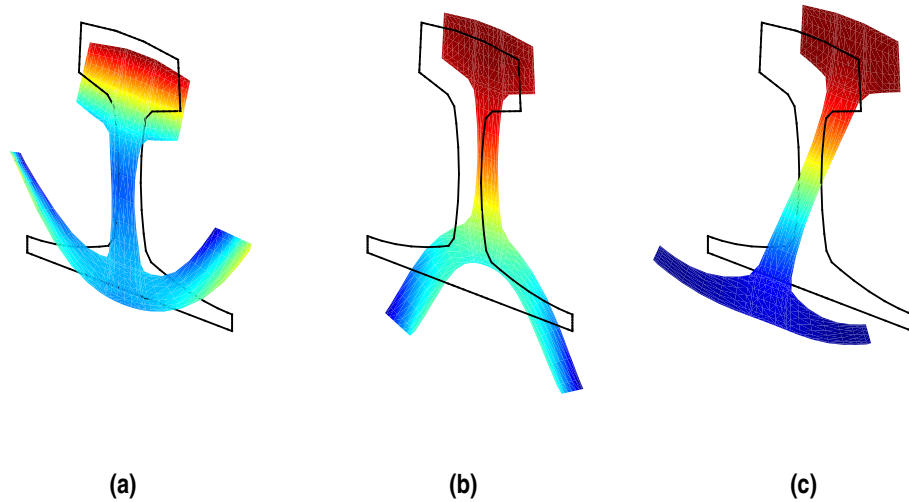


Figure 3.11 Axionometric representation of the cross-section modes of three axial waves at excitation frequency 5600 Hz; (a) $ax1$ wave, (b) $ax2$ wave, (c) $ax3$ wave.

translates horizontally as a rigid body. As the frequency increases, the feet rotation also increases and the neck of the profile starts to deform.

3.2.4 Torsion of the free rail

Figure 3.14 shows the mode of the cross-section due of torsional waves. At low frequencies the modal displacement can be characterized as a rigid body rotation of the cross-section. As frequency increases, the feet starts to vibrate more than the head, which leads to a deformation of the neck. At high frequencies the cross-section presents a S in plane deformation.

3.2.5 Other types of propagative waves

The cross-section mode of propagative wave $c1$ and $c2$ cannot be described by using the analogies with simple beam waves. These two modes are characterized by the rotation of the head, by out-of-phase flapping of the feet and by deformation of the neck. The apparent increase of the dimesion of the hed of the rail is due to the graphical representation of the modal displacement, which are scaled to make the modes visible.

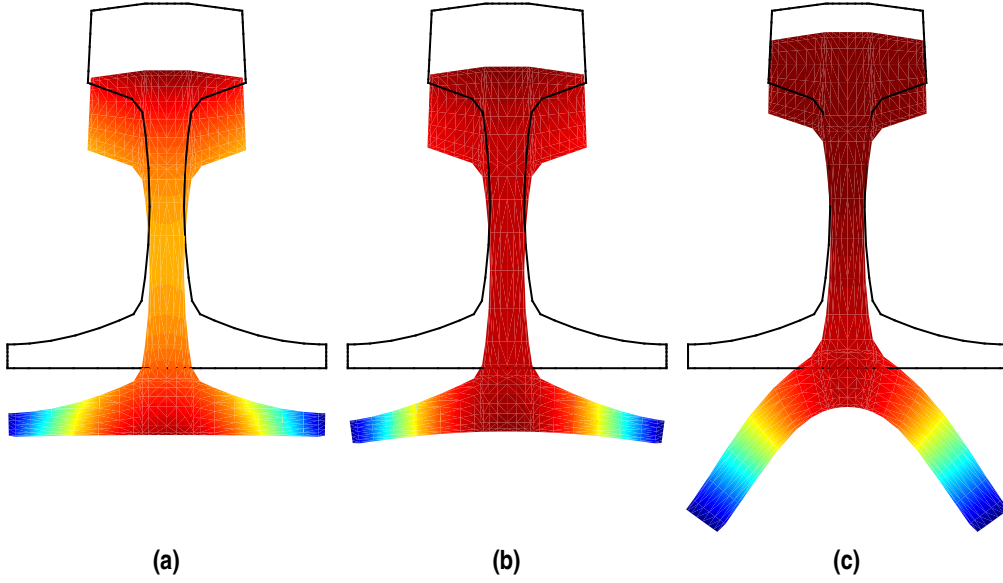


Figure 3.12 Front view of the cross-section mode of the vertical flexural wave fv at excitation frequencies (a) 450 Hz, (b) 1500 Hz and (c) 4200 Hz.

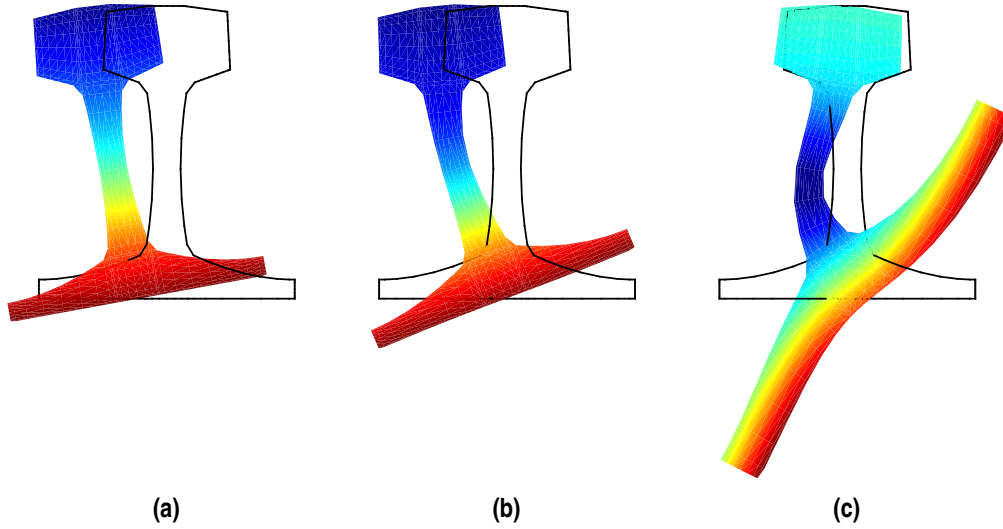


Figure 3.13 Front view of the cross-section mode of the horizontal flexural wave fh at excitation frequencies (a) 450 Hz, (b) 1500 Hz and (c) 4200 Hz.

These results are all in concordance with those described in Reference [10]. Therefore, the accuracy of the proposed method was verified.

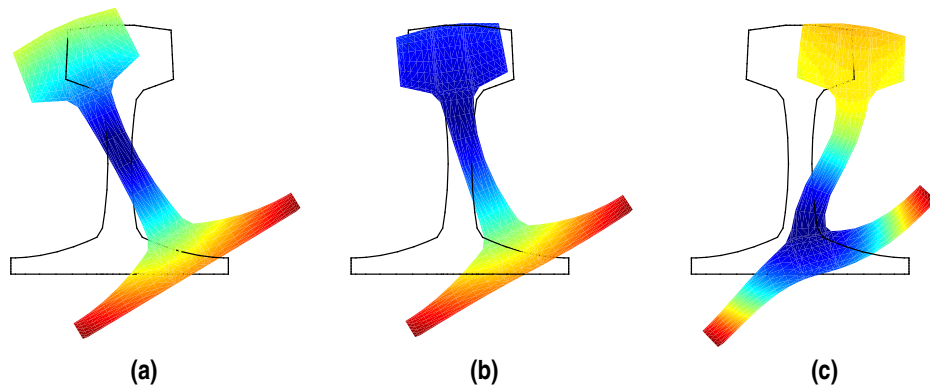


Figure 3.14 Front view of the cross-section mode of the torsional wave to at excitation frequencies (a) 450 Hz, (b) 1500 Hz and (c) 4200 Hz.

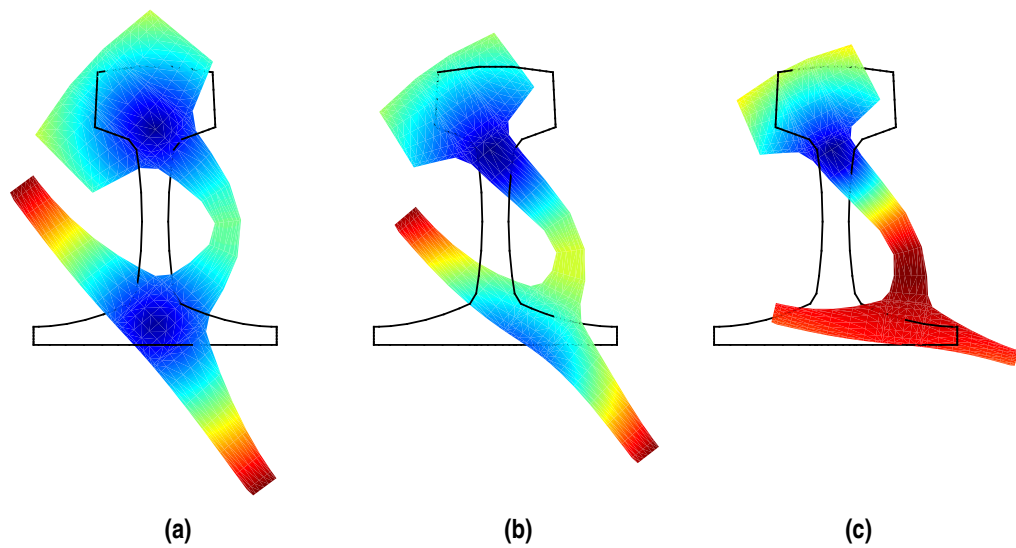


Figure 3.15 Front view of the cross-section mode of the propagative wave $c1$ at excitation frequencies (a) 1500 Hz, (b) 2800 Hz and (c) 4200 Hz.

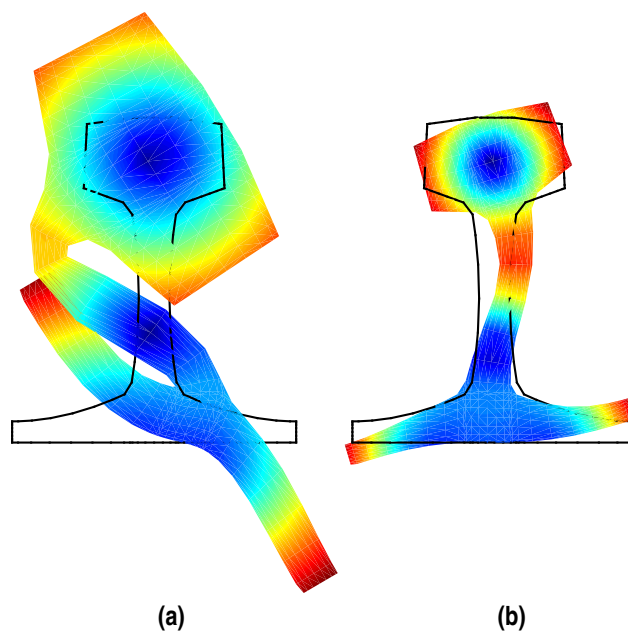


Figure 3.16 Front view of the cross-section mode of the propagative wave c_2 at excitation frequencies (a) 4200 Hz and (b) 5200 Hz.

4 Conclusions and further developement

This work has proposed two spectral based formulations based on the FEM and the BEM to study fluid and solid wave propagation. Both models look at 3D problems whose materials and geometric properties remain homogeneous in one direction. Solid subdomain was modelled with the FEM, whereas the boundary of unbounded fluid media was modelled with the BEM. A spectral 2.5D FEM element for solid media and a spectral 2.5D BEM element for fluid-acoustics media were developed.

The models were verified with two benchmark problems with known solutions. The BEM was verified with a problem consisting of a cavity embedded in an unbounded fluid medium subjected to an incident wave field. The FEM was verified by computation the dispersion curves of propagative waves in free rail. Numerical results were in good agreement with the reference results.

Future researches should complete the work presented herein. The first step is improvement of the BEM model. This can be achieved by solving the singular integrals using more accurate methods, such as closed forms or a regularization procedure. The next step is to complete the coupling procedure in order to study fluid-solid interaction problems. Then, the radiated wavefield in an unbounded fluid medium caused by the deformation of a solid structure due to a point load can be studied.

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Trabajo Fin de Máster
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nales y Puertos

Un método espectral formulado en 2.5D ba-
sado en el MEC y el MEF para estudiar pro-
pagación de ondas en medios acústicos y
elásticos. Resumen

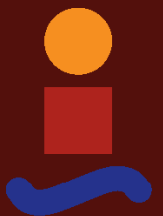
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Resumen

En este trabajo se presenta una formulación espectral en dos dimensiones y media (2.5D) basada en el método de los elementos finitos (MEF) y el método de los elementos de contorno (MEC) para estudiar la propagación de ondas en los ámbitos de la acústica de fluidos y la elastodinámica tridimensionales (3D). El análisis se lleva a cabo mediante la superposición de problemas bidimensionales (2D) para diferentes números de onda a lo largo de la coordenada longitudinal. El método numérico está basado en la descomposición del dominio para representar el sistema fluido-estructura. Se presenta en este trabajo un elemento finito espectral para representar ondas guiadas en sólidos con sección transversal arbitraria. Además, el MEC se extiende a la formulación espectral para estudiar medios fluidos infinitos. Ambas aproximaciones utilizan polinomios de interpolación de Lagrange como funciones de forma en los puntos de Legendre-Gauss-Lobatto (LGL). Las técnicas propuestas se han verificado mediante dos problemas de referencia: la propagación de ondas en un medio fluido infinito en presencia de una cavidad rígida fija y la propagación de ondas en un carril libre, y los resultados obtenidos presentan un buen grado de acuerdo con las soluciones de referencia. Se realiza un breve análisis $h - p$ para evaluar la precisión de los métodos.

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1 Introducción

1.1 Esquema del problema y motivación

La propagación de ondas armónicas en medios acústicos y elásticos es un fenómeno común en muchos campos de la ingeniería. Los métodos híbridos basado en el Método de los Elementos de Contorno (MEC) y en el Método de los Elementos Finitos (MEF) son adecuados para estudiar problemas de interacción fluido estructura. En el rango de frecuencias bajo, los elementos finitos convencionales con funciones de forma lineales representan de forma precisa las ondas radiadas en fluidos y sólidos. Sin embargo, en el rango de frecuencias altas, estas funciones de forma no proporcionan resultados de confianza debido a lo que se conoce como efectos de polución [1, 14]: la precisión de la solución numérica empeora cuando aumenta el número de onda adimensional y no es suficiente la regla de n elementos por longitud de onda [11].

Los efectos de polución (dispersión numérica) no pueden evitarse en problemas bidimensionales y tridimensionales [17], pero pueden reducirse mediante funciones de forma de orden elevado. Ihlenburg y Babška [12, 13] demostraron que formulaciones de orden elevado mejoran la precisión de los métodos numéricos reduciendo las dimensiones de las discretizaciones. Mehdizadeh y Paraschivoiu [16] estudiaron varias metodologías de elementos finitos para la ecuación de Helmholtz. Los autores llegaron a la conclusión de que el p -FEM reduce drásticamente los efectos de polución. Mehdizadeh y Paraschivoiu [16] compararon el Método de los Elementos Espectrales (SEM) con un método MEF de segundo orden. Llegaron a la conclusión que el SEM requiere un menor número de elementos por longitud de onda y un menor coste computacional. Vos et al. [23] presentaron una implementación eficiente del método de los elementos espectrales $h-p$ y analizaron que combinación específica de tamaño de elementos y orden de aproximación minimizaban el coste computacional para un determinado nivel de precisión.

El p -FEM y las metodologías espectrales se han desarrollados usando diferentes familias de funciones de forma: polinomios de Lagrange, Legendre y Bernstein. Biermann et al. [5] y Petersen et al. [19] estudiaron varias familias de funciones de forma para simulaciones acústicas. Los autores concluyeron que el método de los elementos finitos de alto orden are eficiente, incluso para el rango alto de frecuencias, y los elementos basados en los polinomios de Bernstein dan lugar a la solución más eficiente y estable.

Muchos autores han aplicado el metodo de los elementos espectrales en diferentes campos. Degrande y De Roeck [7] desarrollador un método de orden elevado para estudiar la respuesta dinámica de medios porosos saturados usando elementos espectrales. Los autores usaron como funciones de forma en una formulación en desplazamientos la solución analítica para las ecuaciones de Biot restringidas a la propagación de ondas en deformación plana.

Kudela et al. [15] aplicaron elementos espectrales para estudiar propagación de ondas en una barra isotropica y en una viga de Timoshenko. Los autores propusieron esta metodología para detectar daño en estructuras. Los resultados obtenidos fueron comparados con los obtenidos con la metodología convencional de elementos finitos y los medidos en estructuras. Esta comparación resalto la eficiencia de los elementos espectrales y que la metodología refleja el fenómeno bastante bien.

1.2 Objetivos

El método propuesto en este trabajo considera una aproximación en dos dimensiones y media (2.5D) para representar problemas de interacción fluido-estructura. El estudio de ondas radiadas tridimensionales (3D) en

medios fluidos y sólidos requiere el uso de modelos que demandan un alto coste computacional. La solución se vuelve mucho más simple si el medio es homogéneo en una dirección, aunque la fuente dinámica permanezca 3D. Esta situación se conoce como problemas en dos dimensiones y media (2.5D) [6]. Este trabajo presente una formulación espectral en 2.5D basada en el MEF y en el FEM para estudiar la propagación de ondas en problemas de interacción fluido-estructura. El método propuesto es útil para problemas donde el material y las propiedades geométricas son constantes a lo largo de la coordenada longitudinal, y la fuente presenta un comportamiento 3D.

1.3 Contribuciones originales

Los siguientes puntos proporcionan un breve esquema de lo que se percibe que son contribuciones significativas del presente trabajo.

- El método de los elementos espectrales en 2.5D en medios acústicos ha sido extendido a su formulación espectral para el cálculo de propagación de ondas en medios fluidos. Esto introduce un nuevo y preciso método, con un menor coste computacional, para resolver problemas en un rango de frecuencias elevado.
- Por otra parte, un nuevo elemento finito en 2.5D se ha presentado para el estudio de guía de ondas sólidos.

1.4 Organización del texto

El esquema de este trabajo es como se especifica a continuación. En primer lugar, se presenta el modelo numérico. La formulación espectral de elementos finitos en elastodinámica es descrita. Posteriormente se explica el método de los elementos de contornos en su formulación espectral. Por ultimo, los métodos propuestos se verifican mediante la resolución de dos problemas de referencia [20, 10].

2 Modelo numérico

La formulación espectral está basada en la formulación acomplada en 2.5D del MEC-MEF, donde el sólido (Ω_s) se representa mediante el MEF, mientras que la superficie límite (Γ_f) entre el sólido y el fluido ($\Omega_{f\infty}$) se modeliza mediante el MEC (Figure 2.1). En este trabajo, ambas formulaciones han sido desarrolladas por separado, y el proceso de acoplamiento es una línea para futuros desarrollos.

La formulación 2.5D calcula la solución del problema como la superposición de problemas bidimensionales (2D) con diferente número de onda longitudinal, k_z , en la dirección z . Para calcular la solución tridimensional (3D) se utiliza una transformada de Fourier inversa:

$$a(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} \hat{a}(\hat{\mathbf{x}}, k_z, \omega) e^{-ik_z z} dk_z \quad (2.1)$$

donde $a(\mathbf{x}, \omega)$ es la magnitud desconocida (por ejemplo, desplazamientos o presiones), $\hat{a}(\hat{\mathbf{x}}, k_z, \omega)$ es su representación en el dominio de la frecuencia y el número de onda, $\hat{\mathbf{x}} = \mathbf{x}(x, y, 0)$, ω es la frecuencia angular, y $i = \sqrt{-1}$. La integral definida en la Ecuación (2.1) se convierte en una suma de términos de la forma:

$$a(\mathbf{x}, \omega) = \sum_{-\infty}^{+\infty} \hat{a}(\hat{\mathbf{x}}, k_z, \omega) e^{-ik_z z} \Delta k_z \quad (2.2)$$

con $k_z = m\Delta k_z$. Esta ecuación converge y puede aproximarse por una suma finita de términos.

2.0.1 La formulación espectral 2.5D del método de los elementos de contorno

La formulación de los elementos de contornos presentada en este trabajo considera un contorno arbitrario sumergido en un medio fluido infinito. La representación integral de la presión p^i para un punto i localizado en el subdominio fluido $\Omega_{f\infty}$, sin fuerzas de volumen ni condiciones de contorno se escribe como [9]:

$$c^i p^i(\mathbf{x}^i, \omega) = \int_{\Gamma_f} p^{i*}(\mathbf{x}, \omega; \mathbf{x}^i) u^i(\mathbf{x}, \omega) d\Gamma - \int_{\Gamma_f} u^{i*}(\mathbf{x}, \omega; \mathbf{x}^i) p^i(\mathbf{x}, \omega) d\Gamma \quad (2.3)$$

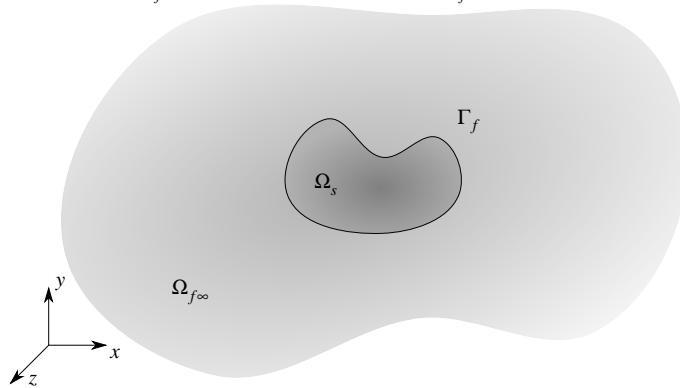


Figura 2.1 Descomposición del dominio.

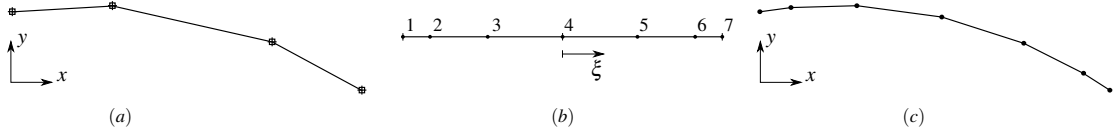


Figura 2.2 (a) Representación de la malla base, (b) definición del elemento en coordenadas naturales y (c) representación en coordenadas físicas del elemento de contorno espectral fluido de orden $p = 6$.

donde $u^i(\mathbf{x}, \omega)$ y $p^i(\mathbf{x}, \omega)$ son respectivamente el desplazamiento normal al contorno Γ_f y la presión nodal. $u^{i*}(\mathbf{x}, \omega; \mathbf{x}^i)$ y $p^{i*}(\mathbf{x}, \omega; \mathbf{x}^i)$ son respectivamente las soluciones fundamentales para un espacio completo fluido para el desplazamiento normal y la presión en un punto \mathbf{x} debido a una carga puntual actuando en \mathbf{x}^i . El término fuera de la integral c^i sólo depende de la geometría del contorno en el punto i . El contorno de integración representa el contorno entre el medio fluido infinito ($\Omega_{f\infty}$) y el subdominio sólido (Ω_s).

A continuación se expone la formulación espectral del método de los elementos de contorno desarrollada en este trabajo.

$$c^i p^i(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} \int_{\Sigma_f} p^{i*}(\mathbf{x}, \omega; \mathbf{x}^i) u^i(\mathbf{x}, \omega) dS dz - \int_{-\infty}^{+\infty} \int_{\Sigma_f} u^{i*}(\mathbf{x}, \omega; \mathbf{x}^i) p^i(\mathbf{x}, \omega) dS dz \quad (2.4)$$

$$c^i \hat{p}^i(\hat{\mathbf{x}}, \omega, k_z) = \int_{\Sigma_f} \hat{p}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i) \hat{u}^i(\hat{\mathbf{x}}, \omega, k_z) dS - \int_{\Sigma_f} \hat{u}^{i*}(\hat{\mathbf{x}}, \omega, k_z; \hat{\mathbf{x}}^i) \hat{p}^i(\hat{\mathbf{x}}, \omega, k_z) dS \quad (2.5)$$

$$c^i \hat{p}^i = \sum_{j=1}^Q \left[\left\{ \int_{\Sigma_f^j} \hat{p}^{i*} \phi^j d\Sigma \right\} \hat{u}^i - \left\{ \int_{\Sigma_f^j} \hat{u}^{i*} \phi^j d\Sigma \right\} \hat{p}^i \right] = \sum_{j=1}^Q [\hat{\mathbf{H}}^{ij} \hat{u}^i - \hat{\mathbf{G}}^{ij} \hat{p}^i] \quad (2.6)$$

El sistema de ecuaciones para todo el contorno de integración queda:

$$\hat{\mathbf{H}}(\omega, k_z) \hat{\mathbf{u}}(\hat{\mathbf{x}}, \omega, k_z) = \hat{\mathbf{G}}(\omega, k_z) \hat{\mathbf{p}}(\hat{\mathbf{x}}, \omega, k_z) \quad (2.7)$$

El método de los elementos de contorno en su formulación espectral para un elemento fluido en 2.5D utiliza polinomios de Legendre de orden p como funciones de forma. Estas funciones ϕ se definen como:

$$\phi_i = \prod_{j \neq i} \frac{\xi - \xi_j}{\xi_j - \xi_i} \quad (2.8)$$

donde el sistema de coordenadas nodales ξ se encuentran en los puntos de Legendre-Gauss-Lobatto:

$$(1 - \xi^2) \frac{\partial \phi(\xi)}{\partial \xi} = 0 \quad (2.9)$$

El contorno Γ_f se discretiza en elementos que son generados a partir de una malla base. La definición de los elementos se hace mediante el uso de polinomios de interpolación de la malla base y su representación en el sistema de coordenadas naturales. La Figura 2.2.(a) muestra la malla base para definir un elemento de orden $p = 6$. En ella se muestra también (b) la representación en coordenadas naturales del elemento, y (c) su representación en coordenadas físicas.

2.0.2 La formulación espectral 2.5D del método de los elementos finitos

La formulación espectral para el subdominio sólido se basa en el principio de los trabajos virtuales. Para un campo de desplazamientos virtuales arbitrario $\delta \mathbf{u}$ impuesto en el sólido Ω_s , el trabajo virtual debido a las fuerzas internas y de inercia debe ser igual al trabajo virtual producido por las cargas externas [25]:

$$\begin{aligned} & -\omega^2 \int_{\Omega_s} \delta \mathbf{u}(\mathbf{x}, \omega) \rho_s \mathbf{u}(\mathbf{x}, \omega) d\Omega + \int_{\Omega_s} \delta \boldsymbol{\varepsilon}(\mathbf{x}, \omega) \boldsymbol{\sigma}(\mathbf{x}, \omega) d\Omega \\ & = \int_{\Omega_s} \delta \mathbf{u}(\mathbf{x}, \omega) \rho_s \mathbf{b}(\mathbf{x}, \omega) d\Omega + \int_{\Gamma_s} \delta \mathbf{u}(\mathbf{x}, \omega) \mathbf{f}(\mathbf{x}, \omega) d\Gamma \end{aligned} \quad (2.10)$$

donde $\mathbf{u}(\mathbf{x}, \omega)$ es el vector de desplazamientos, $\boldsymbol{\varepsilon}$ y $\boldsymbol{\sigma}$ son respectivamente el tensor de deformaciones y el tensor de tensiones, $\rho_s \mathbf{b}(\mathbf{x}, \omega)$ son las fuerzas de dominio Ω_s , ρ_s es la densidad del sólido, y $\mathbf{f}(\mathbf{x}, \omega)$ es el vector de fuerzas nodales. Una variable precedida de δ indica un cambio virtual de su magnitud.

El vector de desplazamientos $\mathbf{u}(\mathbf{x}, \omega)$ se aproxima de la forma:

$$\mathbf{u}(\mathbf{x}, \omega) = \mathbf{N} \underline{\mathbf{u}}(\mathbf{x}, \omega) \quad (2.11)$$

donde \mathbf{N} son las funciones de forma y $\underline{\mathbf{u}}(\mathbf{x}, \omega)$ es el vector de desplazamientos nodales.

El tensor de deformaciones $\boldsymbol{\varepsilon}$ se deriva del vector de desplazamientos nodales $\underline{\mathbf{u}}(\mathbf{x}, \omega)$ como:

$$\boldsymbol{\varepsilon} = \mathbf{L}_1 \mathbf{N} \underline{\mathbf{u}} + \mathbf{L}_2 \mathbf{N} \frac{\partial \underline{\mathbf{u}}}{\partial z} \quad (2.12)$$

donde

$$\mathbf{L}_1 = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial x} \end{bmatrix} \quad (2.13)$$

y

$$\mathbf{L}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (2.14)$$

El tensor de deformaciones $\boldsymbol{\varepsilon}$ se puede reescribir como una combinación lineal de los elementos del vector de desplazamientos, de forma que:

$$\boldsymbol{\varepsilon} = \mathbf{B}_1 \underline{\mathbf{u}} + \mathbf{B}_2 \frac{\partial \underline{\mathbf{u}}}{\partial z} \quad (2.15)$$

donde $\mathbf{B}_1 = \mathbf{L}_1 \mathbf{N}$ y $\mathbf{B}_2 = \mathbf{L}_2 \mathbf{N}$.

El tensor de tensiones está relacionado con el tensor de deformaciones a través de la ley constitutiva del material:

$$\boldsymbol{\sigma} = \mathbf{D} \boldsymbol{\varepsilon} \quad (2.16)$$

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad (2.17)$$

A continuación se expone la formulación espectral del método de los elementos finitos desarrollada en este trabajo.

$$\begin{aligned} & -\omega^2 \int_{\Omega_s} \delta \underline{\mathbf{u}}^T \mathbf{N}^T \rho_s \mathbf{N} \underline{\mathbf{u}} d\Omega + \int_{\Omega_s} \left(\delta \underline{\mathbf{u}}^T \mathbf{B}_1^T + \delta \left(\frac{\partial \underline{\mathbf{u}}}{\partial z} \right)^T \mathbf{B}_2^T \right) \mathbf{D} \left(\mathbf{B}_1 \underline{\mathbf{u}} + \mathbf{B}_2 \frac{\partial \underline{\mathbf{u}}}{\partial z} \right) d\Omega \\ & = \int_{\Omega_b} \delta \underline{\mathbf{u}}^T \mathbf{N}^T \rho_s \mathbf{b} d\Omega + \int_{\Gamma_f} \delta \underline{\mathbf{u}}^T \mathbf{f} d\Gamma \end{aligned} \quad (2.18)$$

$$\begin{aligned}
& -\omega^2 \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{N}^T \rho_s \mathbf{N} dA \right) \underline{\mathbf{u}} dz + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 dA \right) \underline{\mathbf{u}} dz \\
& + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 dA \right) \frac{\partial \underline{\mathbf{u}}}{\partial z} dz + \int_{-\infty}^{\infty} \delta \left(\frac{\partial \underline{\mathbf{u}}}{\partial z} \right)^T \left(\int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 dA \right) \underline{\mathbf{u}} dz \\
& + \int_{-\infty}^{\infty} \delta \left(\frac{\partial \underline{\mathbf{u}}}{\partial z} \right)^T \left(\int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 dA \right) \frac{\partial \underline{\mathbf{u}}}{\partial z} dz \\
& = \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{N}^T \rho_s \mathbf{b} dA \right) dz + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{\Sigma_f} \mathbf{N}^T \mathbf{f} d\Sigma \right) dz
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
& -\omega^2 \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left[\int_{A_s} \mathbf{N}^T \rho_s \mathbf{N} dA \right] \underline{\mathbf{u}} dz + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left[\int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 dA \right] \underline{\mathbf{u}} dz \\
& + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left[\int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 dA - \int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 dA \right] \frac{\partial \underline{\mathbf{u}}}{\partial z} dz \\
& - \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 dA \right) \frac{\partial^2 \underline{\mathbf{u}}}{\partial z^2} dz \\
& = \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{A_s} \mathbf{N}^T \rho_s \mathbf{b} dA \right) dz + \int_{-\infty}^{\infty} \delta \underline{\mathbf{u}}^T \left(\int_{\Sigma_f} \mathbf{N}^T \mathbf{f} d\Sigma \right) dz
\end{aligned} \tag{2.20}$$

$$-\omega^2 \mathbf{M} \underline{\mathbf{u}} + \mathbf{K}^0 \underline{\mathbf{u}} + \mathbf{K}^1 \frac{\partial \underline{\mathbf{u}}}{\partial z} + \mathbf{K}^2 \frac{\partial^2 \underline{\mathbf{u}}}{\partial z^2} = \underline{\mathbf{f}} \tag{2.21}$$

$$\mathbf{M} = \int_{A_s} \mathbf{N}^T \rho_s \mathbf{N} dA \tag{2.22}$$

$$\begin{aligned}
\mathbf{K}^0 &= \int_{A_s} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 dA \\
\mathbf{K}^1 &= \int_{A_s} (\mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 - \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1) dA \\
\mathbf{K}^2 &= \int_{A_s} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 dA
\end{aligned} \tag{2.23}$$

$$\underline{\mathbf{f}} = \int_{A_s} \mathbf{N}^T \rho_s \mathbf{b}(x, y, z, \omega) dA + \int_{\Sigma_f} \mathbf{N}^T \mathbf{f}(x, y, z, \omega) d\Sigma \tag{2.24}$$

La Ecuación (2.21) se transforma al dominio del número de onda como:

$$[-\omega^2 \mathbf{M} + \mathbf{K}^0 - ik_z \mathbf{K}^1 - k_z^2 \mathbf{K}^2] \hat{\underline{\mathbf{u}}}(k_z, \omega) = \hat{\underline{\mathbf{f}}}(k_z, \omega) \tag{2.25}$$

El método propuesto utiliza polinomios de Legendre de orden p como funciones de forma \mathbf{N} , y el sistema de coordenadas nodales $\boldsymbol{\xi}(\xi, \eta) \in [-1, 1] \times [-1, 1]$ se define como los puntos de Lobatto-Gauss-Legendre (LGL):

$$\begin{cases} (1 - \xi^2) \frac{\partial \mathbf{N}(\xi, \eta)}{\partial \xi} = 0 \\ (1 - \eta^2) \frac{\partial \mathbf{N}(\xi, \eta)}{\partial \eta} = 0 \end{cases} \tag{2.26}$$

El subdominio sólido Ω_s se discretiza en elementos que se generan a partir de una malla base. La Figura 2.3.(a) muestra un elemento de orden $p = 6$. La figura también muestra (b) la representación del elemento en coordenadas naturales y (c) físicas.

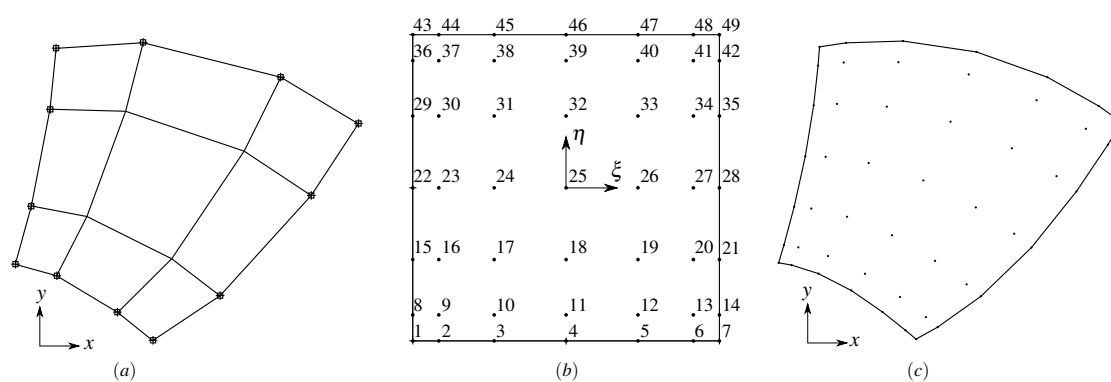


Figura 2.3 (a) Representación de la malla base, (b) definición del elemento en coordenadas naturales y (c) representación en coordenadas físicas de un elemento espectral de orden $p = 6$.

3 Verificación del modelo numérico

Los métodos espectrales de elementos finitos y de contorno han sido verificados mediante la resolución de dos problemas de referencia, una cavidad cilíndrica sumergida en un medio fluido infinito, y la propagación de ondas en un carril libre.

3.0.1 Cavidad cilíndrica en un medio fluido

El modelo MEC se ha verificado con un problema de referencia. El modelo se ha validado mediante la solución de una cavidad cilíndrica, sumergida en un medio fluido homogéneo. La cavidad está sometida a una carga de presión puntual armónica. La solución analítica se puede encontrar en la Referencia [20].

La cavidad tiene un radio $r = 5$ m, y las propiedades del fluido son la velocidad de propagación de las ondas $\alpha = 1500$ m/s y la densidad $\rho = 1000$ kg/m³.

El problema se ha resuelto para un rango de frecuencias de entre 2.5 Hz hasta 320 Hz, con un paso de $\Delta f = 2.5$ Hz. Se ha tomado un valor constante $k_z = 10^{-5}$ rad/m. El campo de presiones radiado se ha calculado mediante la identidad de Somigliana.[9].

La solución del problema se ha calculado para una fuente situada en el fluido en el punto $\hat{\mathbf{x}}_0 = (x_0, y_0) = (0, 15)$ a 15 m del centro de la cavidad.

El tamaño de los elementos y el orden de aproximación se ha establecido para mantener una densidad de nodos por longitud de onda de al menos $d_\lambda = 12$ [4].

Se han considerado tres discretizaciones diferentes, con una longitud de elementos de $1/h = 2/\pi$ m⁻¹, $1/h = 3/\pi$ m⁻¹ y $1/h = 4/\pi$ m⁻¹. En este trabajo se ha tomado un orden de elemento de $p = 5$.

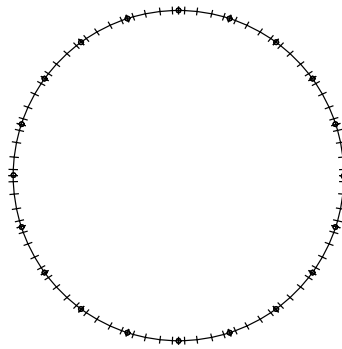


Figura 3.1 Cavidad cilíndrica embebida en un medio fluido infinito: discretización con una distribución de elementos $1/h = 2/\pi$ y orden $p = 6$.

El campo de ondas radiado se ha calculado en una malla de receptores como se muestra en la Figura ??.

La Figura 4.3 muestra la amplitud de la solución analítica del campo de presiones radiado y el campo total de presiones cuando actúa una presión puntual de 200 Hz. La Figure 4.4 representa el error absoluto entre los resultados numéricos y la solución analítica para las diferentes discretizaciones estudiadas.

Los máximos errores aparecen en la zona de sombra y alrededor del contorno de la inclusión. La solución numérica mejora cuanto mayor sea la distancia al contorno. Como era de esperar, la precisión del modelo

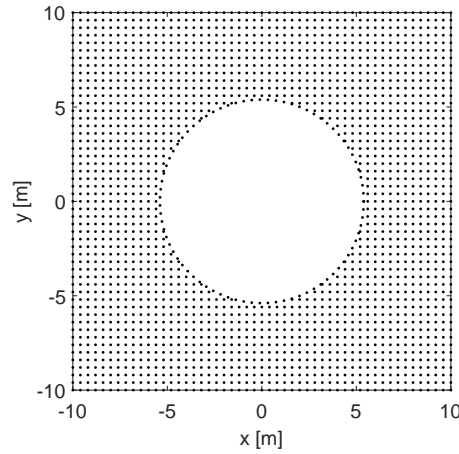


Figura 3.2 Malla de receptores.

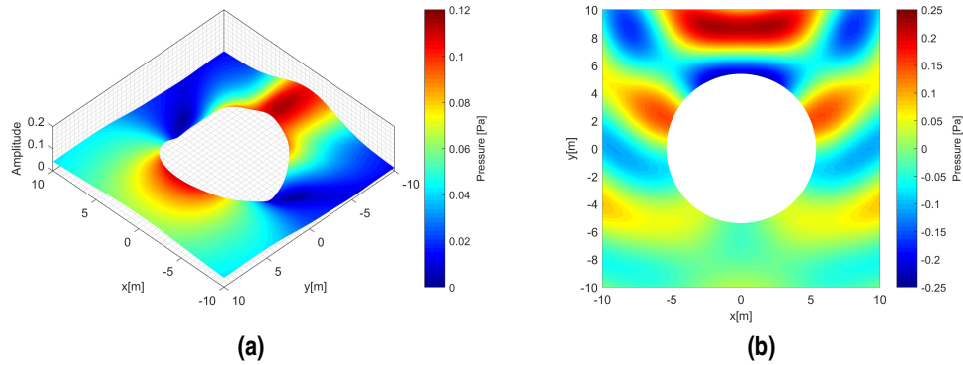


Figura 3.3 (a) Campo de presiones radiado y (b) campo total de presiones debido a una carga armónica actuando a 200 Hz.

mejora al disminuir el tamaño de los elementos. Los resultados muestran que un orden de aproximación mayor o un menor tamaño de elementos debe usarse para calcular la solución en puntos cercanos al contorno.

3.0.2 Propagación de ondas en un carril libre

El elemento sólido espectral presentado en la Sección 2.0.1 se ha verificado mediante el cálculo de las ondas propagativas en un carril libre UIC861-3. Este problema ya ha sido estudiado por L. Gavric [10]. Las curvas de dispersión y los modos de vibración se han calculado para un rango de número de onda $0 - 35$ rad/m, con un paso de $\Delta k_z = 0.05$ rad/m.

La sección transversal se ha representado mediante siete elementos, tres para la cabeza del carril, tres para el patín y uno para el alma. Esta es una ventaja del método de los elementos finitos espectrales, que permite discretizar el dominio con un menor número de elementos. El problema ha sido resuelto con diferentes órdenes de aproximación para evaluar la precisión del método.

Los resultados muestran que aparecen ocho modos de vibración característicos en el rango de frecuencias $0 - 6$ kHz. Cuatro modos tienen componentes de desplazamiento axiales/verticales (Figure ??), mientras que el resto se caracterizan por un movimiento lateral de la sección transversal (Figure ??).

Los resultados numéricos muestran un buen grado de acuerdo con los resultados de referencia para casi todas las ondas descritas. Las ondas de movimiento lateral son muy parecidas a la solución de referencia. Por otra parte, las ondas $ax1$, $ax2$ y $ax3$ muestran diferencias apreciables con los resultados presentados en la Referencia [10] para el rango de frecuencias más elevado. Estas diferencias podrían ser explicadas por la malla gruesa utilizada en la Referencia [10] para frecuencias tan altas. Los resultados propuestos sugieren que la solución para mallas finas convergen a la solución del SEM que aparece en las Figuras 4.6 y 4.7.

Estos resultados están en concordancia con los descritos en la Referencia [10]. De modo que la precisión del método propuesto queda verificada.

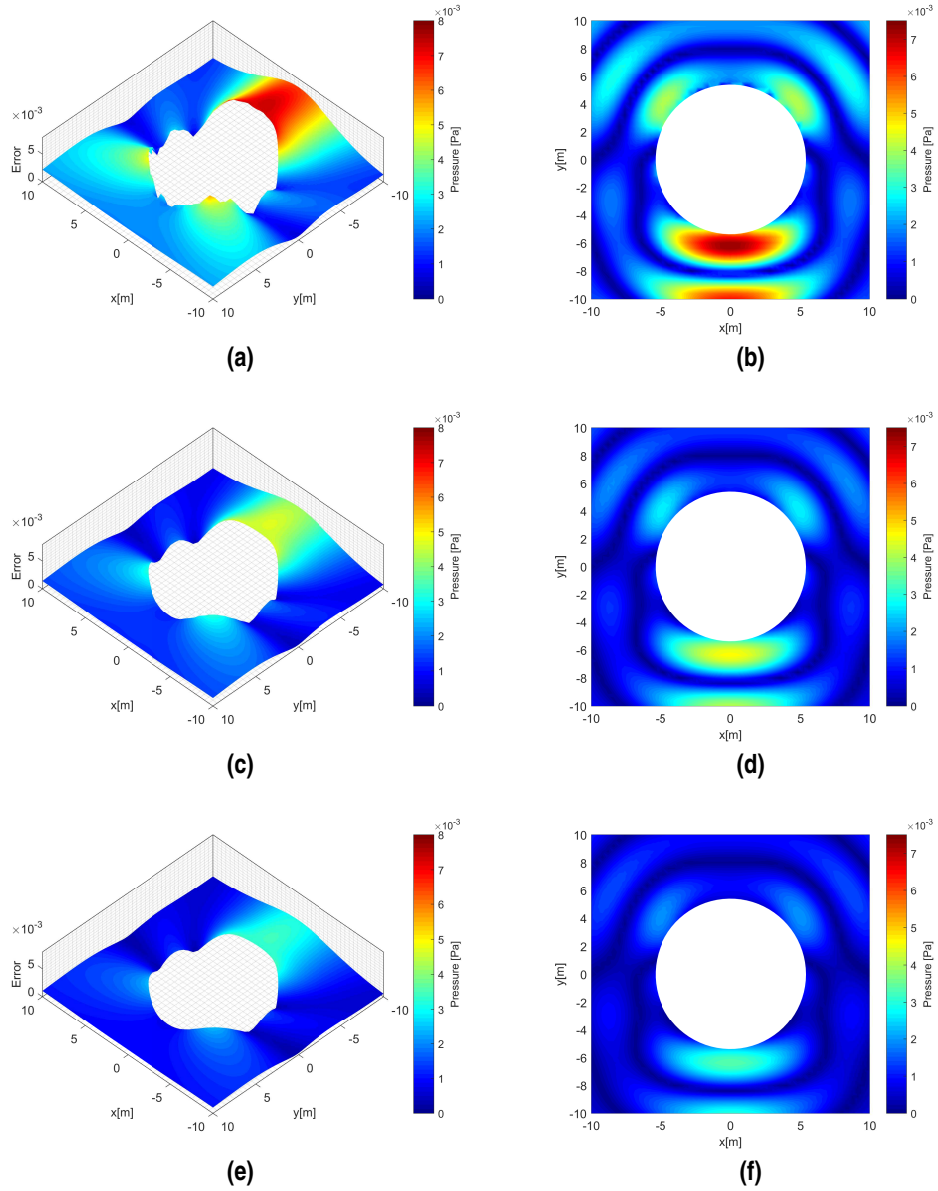


Figura 3.4 Error a

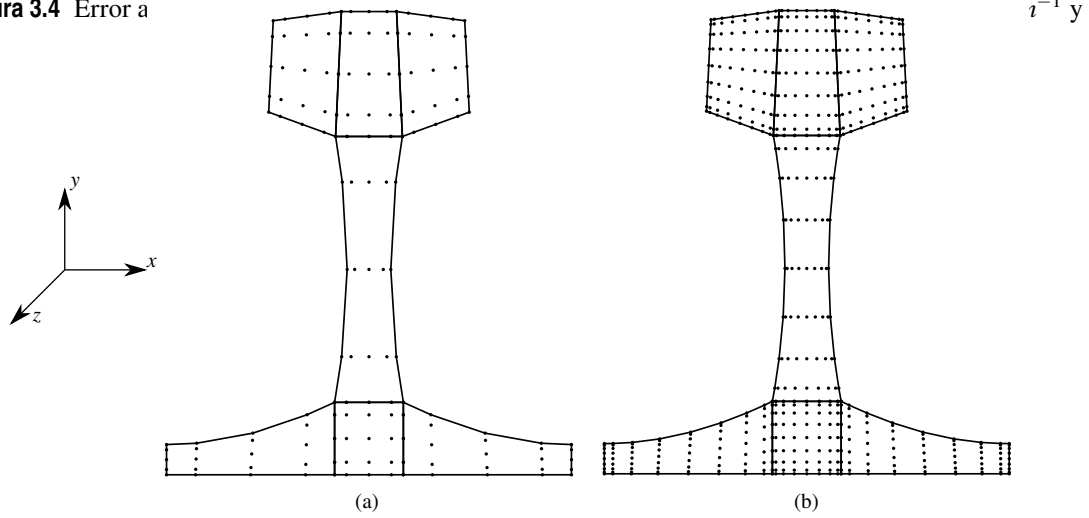


Figura 3.5 Discretización del carril con elementos espectrales. Orden de elementos (a) $p=4$ y (b) $p=8$.

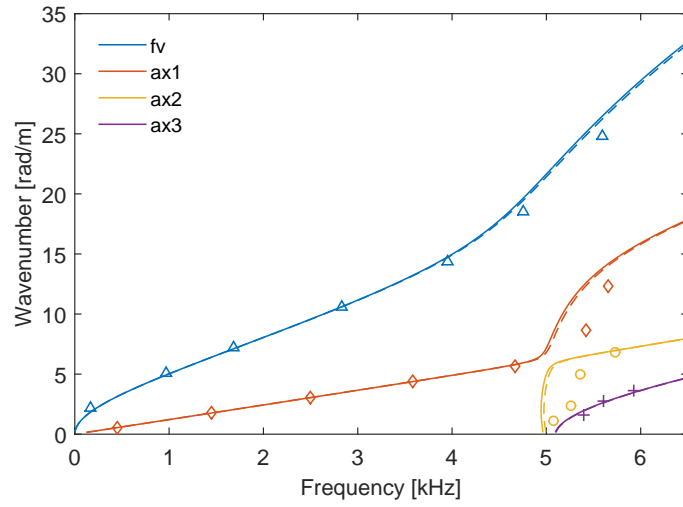


Figura 3.6 Curvas de dispersión de los modos axiales/verticales para el rango de frecuencias 0-6000 Hz. —, resultados numéricos para $p = 8$; - -, resultados numéricos para $p = 4$; \triangle , \diamond , \circ , $+$, resultados de referencia [10].

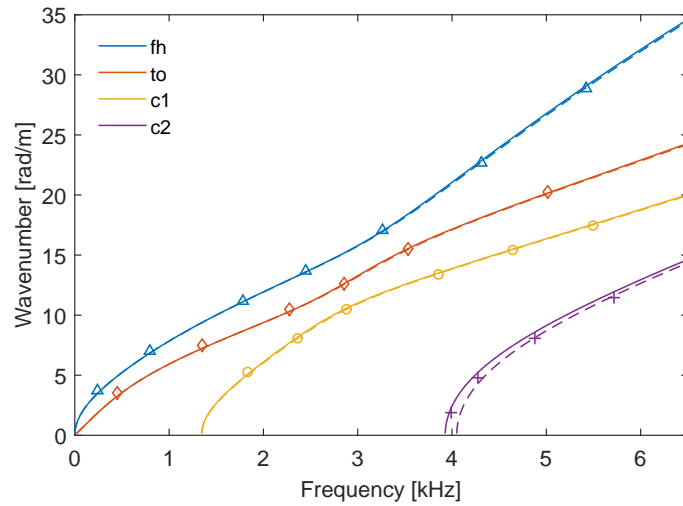


Figura 3.7 Curvas de dispersión de los modos laterales para el rango de frecuencias 0-6000 Hz. —, resultados numéricos para $p = 8$; - -, resultados numéricos para $p = 4$; \triangle , \diamond , \circ , $+$, resultados de referencia [10].

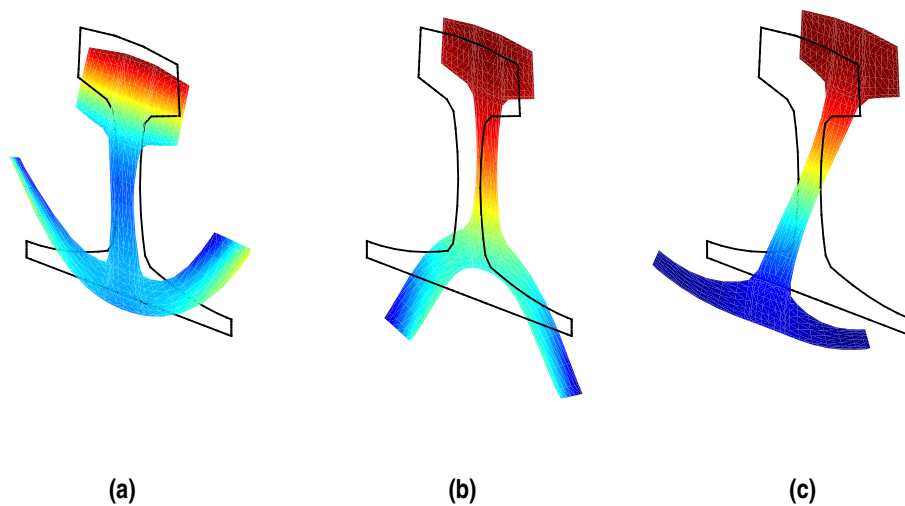


Figura 3.8 Vista frontal del modo de la onda axiales a la frecuencia de 5600 Hz; (a) $ax1$, (b) $ax2$, (c) $ax3$.

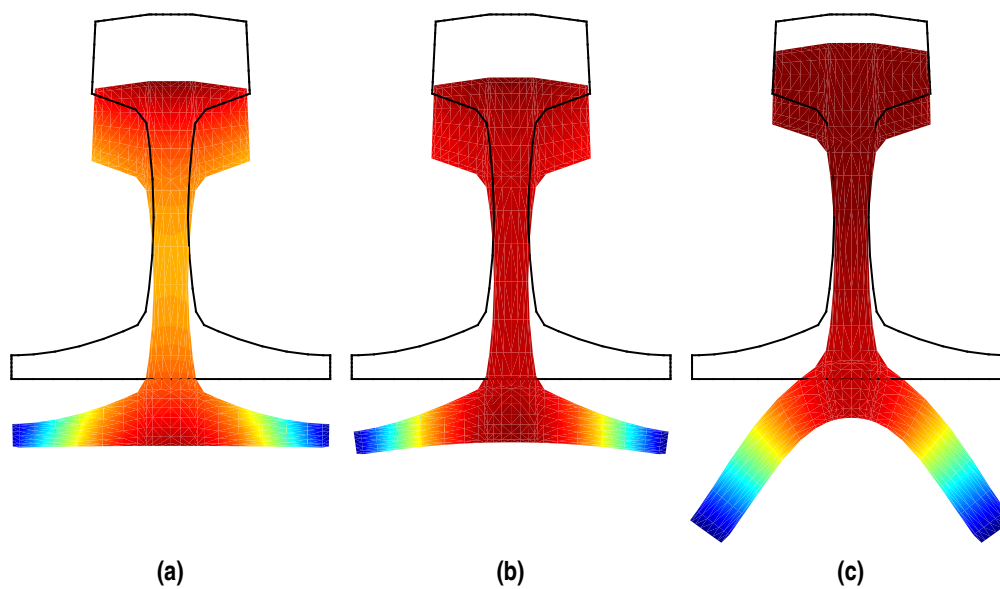


Figura 3.9 Vista frontal del modo de la onda fv a las frecuencias (a) 450 Hz, (b) 1500 Hz y (c) 4200 Hz.

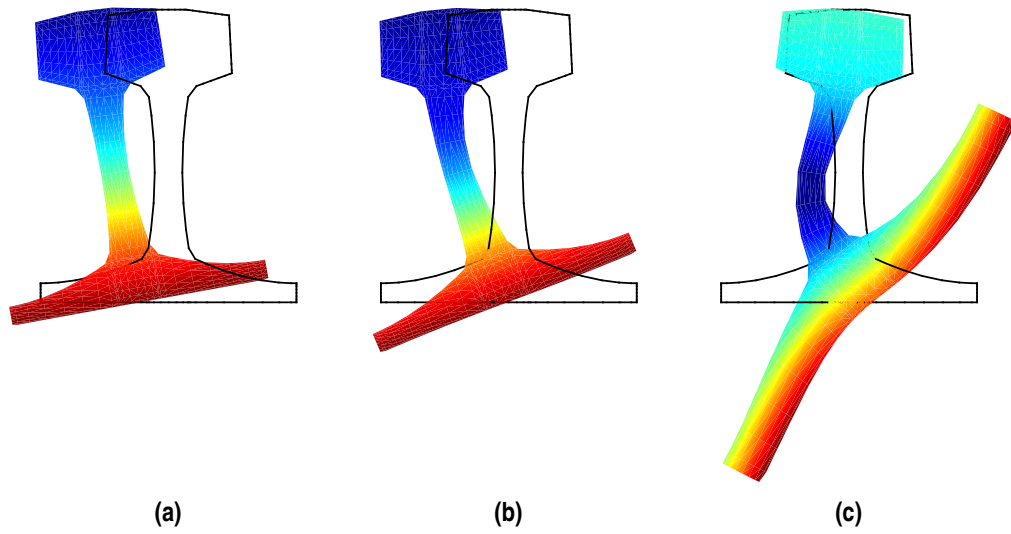


Figura 3.10 Vista frontal del modo de la onda *fh* a las frecuencias (a) 450 Hz, (b) 1500 Hz y (c) 4200 Hz.

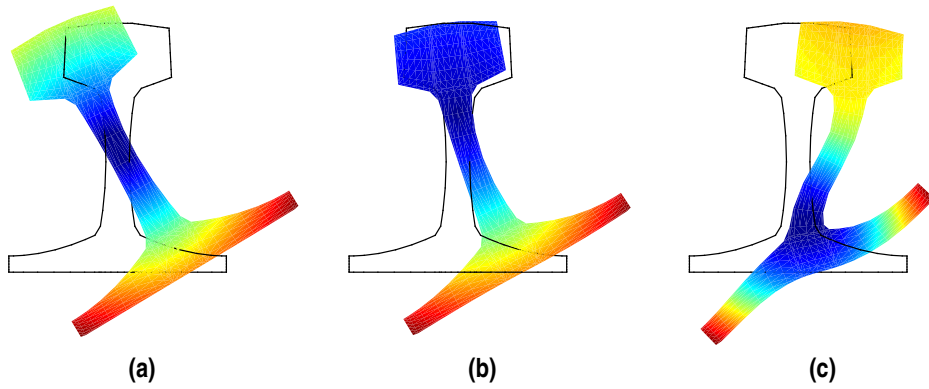


Figura 3.11 Vista frontal del modo de la onda *to* a las frecuencias (a) 450 Hz, (b) 1500 Hz y (c) 4200 Hz.

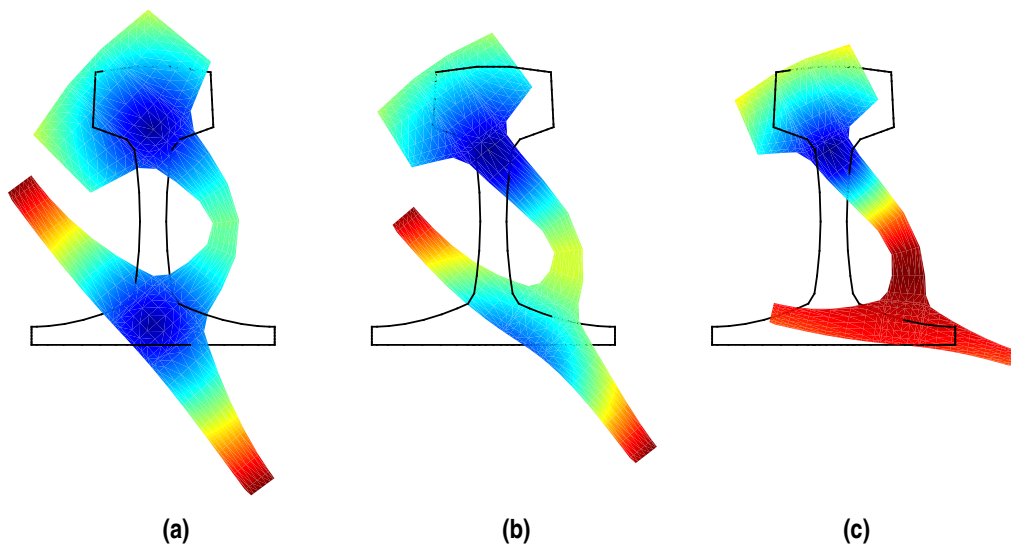


Figura 3.12 Vista frontal del modo de la onda *c1* a las frecuencias (a) 1500 Hz, (b) 2800 Hz y (c) 4200 Hz.

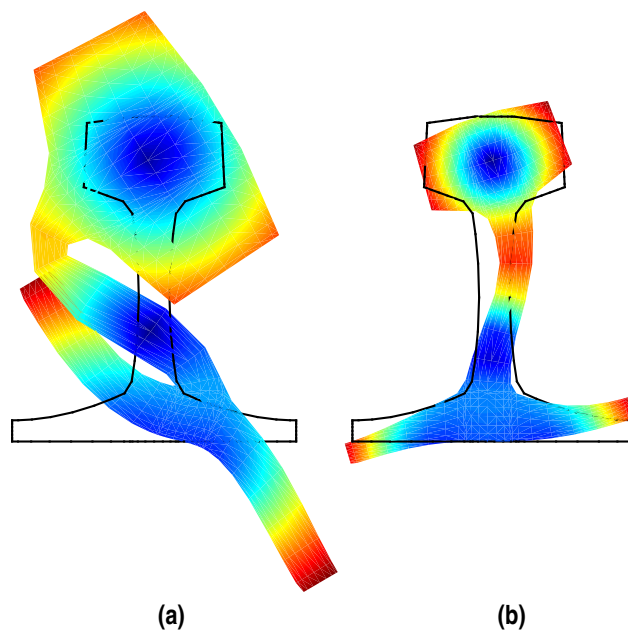


Figura 3.13 Vista frontal del modo de la onda $c2$ a las frecuencias (a) 4200 Hz y (b) 5200 Hz.

4 Conclusiones y desarrollos futuros

En este trabajo se han propuesto dos formulaciones espectrales basadas en el MEC y en el MEF para estudiar propagación de ondas. Ambos modelos resuelven problemas tridimensionales con propiedades del material y geometría constante a lo largo de la coordenada longitudinal. Los subdominios sólidos se representan mediante el MEF, mientras que el contorno del medio fluido se modeliza con el MEC. Los elementos finitos y de contorno espectrales en 2.5D han sido desarrollados.

Los modelos han sido verificados con dos problemas de referencia, de soluciones conocidas. El MEC ha sido validado mediante un problema consistente en una cavidad embebida en un medio fluido infinito sometido a un campo de presiones incidente. El MEF se ha verificado mediante el cálculo de las curvas de dispersión de ondas propagativas en un carril libre. Los resultados numéricos presentaron un buen grado de acuerdo con los resultados de referencia.

Los desarrollos futuros debería completar el trabajo aquí presentado. El primer paso es la mejora del modelo MEC. Esto puede alcanzarse resolviendo las integrales singulares con métodos más precisos, como resolviéndolas analíticamente o mediante un proceso de regularización. El siguiente paso consiste en el proceso de acoplamiento de ambos métodos para estudiar problemas de interacción fluido-estructura. Se podrá estudiar el problema del campo radiado en un fluido infinito causado por la deformación de una estructura debido a una carga puntual.

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